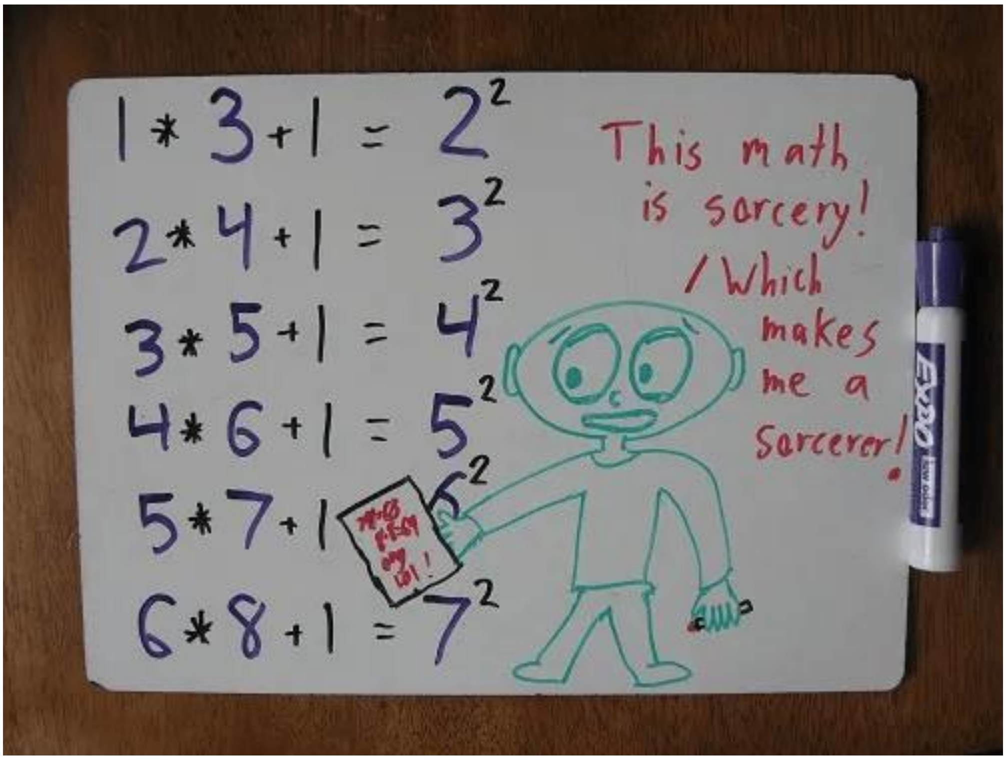
# 1#Thinking like a mathematician

# 1.1 Didactic commentary

*Ann Kiefer*

*To be honest, mathematicians don't do much. They drink coffee, frown at chalkboards. Drink tea, frown at students' exams. Drink beer, frown at proofs they wrote last year and can't for the life of them understand anymore. It's a life of drinking, frowning, and, most of all, thinking. You see, there are no physical objects in math: no chemicals to titrate, no particles to accelerate, no financial markets to destroy. Rather the verbs of the mathematicians all boil down to actions of thought. [...] mathematics is an action of the mind* (B. Orlin, 2018). This is the description of mathematicians and/or mathematics given by B. Orlin (2018) in his book *Math with Bad Drawings: Illuminating the Ideas That Shape Our Reality*.




*Math with Bad Drawings:* https://mathwithbaddrawings.com/2014/01/13/undiscovered-math/

This module gives students the chance to immerse themselves in the role of a mathematician and explore the mindset of mathematical thinking. But what exactly is mathematical thinking?

Music teachers challenge students to listen and participate.

English and History teachers invite students to journey in other worlds.

Art and Drama teachers offer students opportunities to explore.

What are we to offer students if they are to function mathematically? (Gilderdale, 2011)

Unfortunately, many people have a misconception of mathematics, reducing it to simple calculations or multiplication of numbers. In reality, mathematics is a deeply creative discipline that relies more on reflection and written expression than on the complex arithmetic that the general public generally associates with it.

An international research team (Petozc et al. 2007) wanted to find out what first-year university students understood by the term 'mathematics'. The survey was conducted in Australia, Brunei, Canada, Northern Ireland and South Africa (countries with historical links to the British Empire and education systems derived from the British system). Students from mathematics, engineering and computer science courses were surveyed. The first result: more than half the students defined mathematics primarily as a tool for finding solutions to problems. The authors conclude that there is an urgent need to broaden the scope of mathematics and to give students a broader image of mathematics. The researchers suggest that teachers should not focus solely on techniques in lessons and classrooms, but should show how certain professions use and understand mathematics (math teachers, statisticians, engineers, programmers, etc.). For these professions too, numbers and toolboxes are an integral part of mathematics. But this is just the beginning for them.

We've set ourselves the goal of starting this openness to mathematics earlier than university. One of the aims of this module is to show what mathematical thinking, problem solving and proof writing really is, to show how to do these things, and to show how much fun it is. In this module we give (try to give) an answer to this question by breaking with the classic structure of a math lesson, where the teacher explains the theory, does an example on the board in front of the whole class and then the students repeat the same example several times in a 'drill' version. Here, the students are given as a starting point a few examples that illustrate a mathematical curiosity. Then it's up to the students to find more examples themselves and to identify a recurring pattern behind these examples to then formulate a conjecture. Once they have formulated their conjecture, they must try to prove it or find a counterexample. All this work is done in groups, followed by pooling and class discussions. The leitmotif here would be *HOTS not MOTS*, where HOTS stands for *High Order Thinking Skills* and MOTS for *More Of The Same*. This activity will give students the authentic experience of (thinking like) a professional mathematician. Unlike traditional exercises, they will have to explore, analyse and attempt to prove or disprove a theorem based on intriguing observations made previously.

The module "Thinking like a mathematician" should be used in the *algebra* chapter figuring in the official Luxembourgish school programme for 6e (MENJE, 2023) and *aims to model a situation using an algebraic expression*. This activity stimulates pupils' ability to ask themselves the right questions and to "think mathematically". The concept of mathematical thinking is not new. Pólya was one of the first mathematicians to describe how to solve problems mathematically (Pólya, 1945). Schoenfeld (1985, 1992) then extended these fundamental ideas by identifying several components of mathematical thinking: the knowledge base, problem-solving strategies, metacognition (control and self-regulation) and belief systems. To put it in the words of Stein and Smith (1998):

*Tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for student thinking.*

David Baker, professor of sociology and education at the University of Pennsylvania, agrees:

*Overall, there's a movement towards more complex cognitive mathematics, there's a movement towards the student being invited to act like a mathematician instead of passively taking in math and science.* (Hartnett, 2016).

This is not, of course, an easy exercise, either for the student or for the teacher. The difficulties in understanding, making connections and explaining that students will encounter during this activity are referred to by Schoenfeld (2023) as *positive struggle*. This is a mechanism for developing a deep and meaningful understanding of mathematical content.

Teachers need to be curious, intrigued and keen to find out more. However they also have to accept messy work, value risk-taking and half-formed ideas, give time to reflect and encourage discussion. (Gilderdale, 2011). The most important thing, however, is that teachers choose the right topics and groups of pupils so that everyone can benefit from the activity and develop their mathematical thinking skills.

The module "Thinking like a mathematician" is also a good example of a TRU activity. TRU stands for Teaching for Robust Understanding and is a framework founded by Alan H. Schoenfeld, a professor at the University of California. It describes the properties of classrooms in which students become powerful and autonomous thinkers in mathematics (Schoenfeld, 2023). The five dimensions of classroom activity are at the heart of TRU. Classrooms that do well in these 5 dimensions produce students who are powerful thinkers:

**The Five Dimensions of Powerful Classrooms**

| The Content | Cognitive Demand | Equitable Access to Content | Agency, Ownership, and Identity | Formative Assessment |
|---|---|---|---|---|
| *The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers. Discussions are focused and coherent, providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind.* | *The extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called "productive struggle."* | *The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core disciplinary content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.* | *The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations about disciplinary ideas, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.* | *The extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.* |

*The TRU framework in brief, https://truframework.org/*

In short, the aim of this module is to show students the abstract, but also the creative and curious, side of mathematics. We hope to change the image of mathematics as boring, pointless and far removed from reality, to a more positive and exciting one.

*I don't expect, and I don't want, all children to find mathematics an engrossing study, or one that they want to devote themselves to either in school or in their lives. Only a few will find mathematics seductive enough to sustain a long term engagement. But I would hope that all children could experience at a few moments in their careers...the power and excitement of mathematics...so that at the end of their formal education they at least know what it is like and whether it is an activity that has a place in their future.* (Wheeler, D. (1975), quoted in Cross, K. (2004))

Some will criticise this module for being too abstract and not showing the applied side of mathematics. To these we reply that other modules in this PITT already do this and, more importantly, we insist that *mathematics is not only taught because it is useful. It should be a source of delight and wonder, offering pupils intellectual excitement and an appreciation of its essential creativity* (Cross, 2004).

## References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# 1.2 Lesson planning

## 01 Conditions of unit

| | |
|---|---|
| Target audience : | 6e |
| Place : | A typical classroom |
| Materials required: | At least one computer or tablet with Internet access per group of 2 pupils. |
| Duration : | 2 or 3 times 90 minutes |

## 02 Targeted skills

### Skills targeted by the mathematics course

- Use an algebraic expression to model situations and describe dependencies.
- Describe a series of figures or a series of numbers using an algebraic expression.
- Use a calculator or software to calculate a numerical value.
- Decode an algebraic expression into an ordered series of calculation instructions and formalise the algorithm.

### Skills targeted by the natural sciences course (Naturwissenschaften)

- Formulate well-founded hypotheses based on the context, using examples.
- Make targeted observations, present and evaluate the results, and draw conclusions.
- Recognise simple cause-and-effect relationships and draw logical conclusions.
- Diagrammatic representation and interpretation of the models presented.

### Competences covered by the Reference Guide for Media Literacy[1]

- Competences 1: 1.2 Analysing and assessing data, information and digital content.
- Competences 2: 2.1 Working with others 2.3 Using appropriate forms of expression (netiquette).
- Competences 3: 3.3 Modelling, structuring and coding.

[1] https://edumedia.lu/wp-content/uploads/2024/12/Medienkompass_EN_web.pdf

# 03 Over the course of the lesson

In this module, students are required to immerse themselves in mathematical conjectures. They must first explore examples, then establish a conjecture, try to prove it and finally formulate a theorem. The various conjectures are available in a blanc version (M1) (without help or hints) and a guided version (M2) (with help, hints and gap-fill texts).

**Before the lesson**: the teacher divides the pupils into groups of 4. As the problems have different levels of difficulty, it is preferable to group the pupils according to level. The teacher will then decide in advance which group will work on which conjectures.

Note that conjectures 7 and 8 are very difficult. They are supposed to be given to groups of students who are very strong in mathematics. The teacher should also bear in mind that it is not at all necessary to work on every conjecture in this module. If necessary, several groups can also work on the same conjecture.

There are 15 conjectures in all. The conjectures are of different types:

- Conjectures 1 to 8: These are true theorems that are easy to prove and have short proofs. However, they have different levels of difficulty (see table below).
- Conjectures 9 to 12: these are refuted conjectures, i.e. they can be proved false, or a counterexample can be found. These statements are made in such a way that if we restrict ourselves to testing the first 5 cases, these statements appear to be true. However, by testing a larger number of cases (n=40 for 9, the 8$^{th}$example for 10, n=5 for 11 and n=15 for 12), we quickly come across a counterexample that refutes the conjecture.
- Conjectures 13 to 15: These are open conjectures, which mathematicians are convinced are true, but no one has yet been able to prove them. Of course, students are not expected to prove these open conjectures. The idea is to familiarise them for the first time with the concept of an open mathematical conjecture. The conjectures chosen are all easy to understand and the examples illustrating them are feasible for students in secondary school.

The following table summarises the nature and level of difficulty of the various conjectures.

| Nature of the conjecture | Level of difficulty | Easy | Medium | Difficult | Very difficult |
|---|---|---|---|---|---|
| True theorem | | 01,02 | 03, 04, 06 | 05 | 07, 08 |
| Refuted conjecture | | 09, 12 | 10 | 11 | / |
| Open conjecture | | / | 14 | 13, 15 | |

The unit can be done in two or three stages. In the case of two stages, true and provable statements (conjectures 1 to 8) are dealt with during stage 1. In the next stage, open and disprovable statements (conjectures 9 to 15) are dealt with. In other case, step 1 deals with true and provable statements (conjectures 1 to 8), step 2 with disproved statements (conjectures 9 to 12) and the final step with open statements (13 to 15).

## First hour of teaching

**Introduction (15 min)**: The teacher gives a short introduction to the concept of conjectures. The following definitions can be given (or any other definition of conjecture).

**Definition 1**: A **conjecture** is an assertion for which a proof is not yet known, but which is strongly believed to be true.

**Definition 2.** A **conjecture** is a hypothesis that has not yet been confirmed.

**Definition 3.** A **conjecture** is a proposition that is made on the basis of observations, apparent patterns, or numerical results, but which has not yet been formally proven. Mathematicians formulate conjectures when they notice regularities in the data or patterns in their research.[2]

The conjectures are distributed to the students under the form of the blanc version and they work individually for the moment. The teacher tells each pupil which conjecture (from conjectures 1 to 8) to work on (according to the distribution established earlier) and lets them work individually.

**Examples and conjectures (30 min):** The pupils get into groups, the groups being designated by the teacher. They think about examples, a conjecture and an idea for a proof. The teacher moves around the classroom to help and answer questions, if necessary.

## Second hour of teaching

**Formulation of evidence (35 min):** The pupils try to formulate a proof, or an explanation for the phenomenon observed. They think about how to formulate their conclusion.

**Important notes:**

- Students may use any language to discuss in their group. However, the written material must be produced in French. This is important so that, on the one hand, the students practise mathematical writing in French and, on the other hand, by writing in French, the students remain consistent with the language of the course and the vocabulary learned in class.
- In this module, it is important that students dare to think about mathematical problems. Their writing does not have to be perfect. The teacher should encourage them to try to describe their ideas, even if their writing is not mathematically neat and rigorous. The slogan *Promote imperfection to ultimately demand perfection* applies.
- This first part can be done without the use of ICT (Information and Communication Technologies). However, we recommend that teachers allow students to at least use calculators or spreadsheets. For teachers who want to incorporate more technology, other tools may prove useful.

**Conclusion (10min):** Each group will briefly present its conjecture. A group discussion will follow on the complexity of the problem and the key stages (finding examples, formulating a conjecture, finding a proof, formulating a theorem).

**End of lesson:** The teacher will collect all the completed worksheets and correct them for the next time.

[2] Definition taken from the following article https://www.futura-sciences.com/sciences/definitions/mathematiques-conjecture-373/

**Differentiation:** The different conjectures are available in a blanc version (M1) (without help or clues) and a guided version (M2) (with help, clues and fill-in-the-blanks). When a group of pupils seems to be struggling, the teacher can give them the guided version of the problem. The distribution of the guided version will be adapted to the specific needs of each group, depending on their progress and particular difficulties.

Note that the lesson plan is based on the time needed when the students are working on the blanc version of the conjectures. If the whole class works on the guided version from the start, the time needed can be halved.

## Third hour of teaching

**Distribution of corrected worksheets (10 min):** The corrected worksheets of the previous lesson are handed out to the students. The students look through them and can ask questions.

**Introduction (10 min)**: The teacher tells each student which conjecture to work on (among conjectures 9 to 15 if the module is done in two stages only, otherwise among conjectures 9 to 12 and according to the distribution established earlier) and lets them work individually. If the module is done in two stages only, the teacher ensures that at least one conjecture from each category (open statements and disproved statements) is worked on.

**Examples and conjectures (25 min):** The pupils get into groups; the groups being designated by the teacher. They think about examples, a conjecture and an idea for a proof. The teacher circulates around the class to help and answer questions, if necessary.

## Fourth hour of teaching

**Formulating evidence (35 min):** The pupils try to formulate a proof, or an explanation for the phenomenon observed. They think about how to formulate their conclusion.

**Important information:**

- Students may use any language to discuss in their group. However, the written material must be produced in French. This is important so that, on the one hand, the students practise mathematical writing in French and, on the other hand, by writing in French, the students remain consistent with the language of the course and the vocabulary learned in class.
- In this module, it is important that students dare to think about mathematical problems. Their writing does not have to be perfect. The teacher should encourage them to try to describe their ideas, even if their writing is not mathematically neat and rigorous.
- For this second part we recommend the use of ICT (Information and Communication Technologies). Spreadsheets (e.g. Excel) or computer programs (e.g. WolframAlpha, Scratch etc.) are very useful. Without them, counterexamples to statements 9 to 12 are hard to find. For open conjectures, it is necessary to let the students search the Internet or ask an artificial intelligence (e.g. ChatGPT, ShulKI etc.) for help. They could be left to fend for themselves without any research tools at first and only use the Internet later. The fact of thinking about the problem first and only later finding, via a search tool, that the statement is in fact an open conjecture, will increase the pupils' astonishment.

**Conclusion (10 min):** Each group briefly presents its conjecture. A group discussion follows on the complexity of the problem and the key stages (finding examples, formulating a conjecture, finding a counterexample or discovering that it is an open problem, formulating a conclusion). The concepts of counterexample and open problem are discussed.

**End of lesson:** The teacher will collect all the copies and mark them for the next time.

**Differentiation:** The different conjectures are available in a blanc version (without help or clues) and a guided version (with help, clues and gap-fill texts). When a group of pupils seems to be struggling, the teacher can give them the guided version of the problem. The distribution of the guided version will be adapted to the specific needs of each group, depending on their progress and particular difficulties.

Note that the lesson plan is based on the time needed when the students are working on the blanc version of the conjectures. If the whole class works on the guided version from the start, the time needed can be halved.

## Fifth and sixth teaching hours (optional)

These two hours of teaching are optional. If these lessons are done, statements 9 to 12 are dealt with in the third and fourth lessons and statements 13 to 15 in the following two lessons. The same lesson plan applies as before.

## Seventh (resp. Fifth) lesson: Closure

**Distribution of corrected worksheets (10 min):** The corrected worksheets of the previous lesson are handed out to the students. The students look through them and can ask questions.

As described in the previous paragraph, the closure can be done after the fourth or sixth lesson. This lesson takes place according to the concept of *pedagogical walls* (*murs pédagogiques*, Agostino & de Versailles, 2024). Since it was set up at the start of the 2017 academic year at the Lycée de la Plaine de Neauphle in Trappes in the Yvelines, the pedagogical device described in this chapter, which we call "Pedagogical Walls", can now be found in the classroom practices of several teachers. The name comes from the fact that some classrooms are equipped with whiteboards mounted on the walls. The boards are positioned all along the free walls and make it possible to create mini-classes by organising the students' tables in a horseshoe shape around each board.

**Set-up (5 min):** At the start of the session, the pupils are divided into groups of four to six. Each group sits facing one of the boards or simply around a table, as the boards are not essential. It is important to mix the groups and not to keep the same groups as in the previous phases.

*Important note:* If the second and third lessons have been done in one lesson, the teacher must ensure that each group has at least one student who has previously dealt with a false conjecture and one student who has dealt with an open conjecture.

**First phase (18 min).** The fake Instagram publications (M3) are distributed among the different groups. There must be at least two different publications for the second phase of the scenario to be possible. Each group can use the board to share and test their attempts. If boards are not available, students can simply write on sheets of paper or tablets. No write-up is expected. Indeed, it might be tempting to ask the students to write down a solution in their own notebooks in a bid to maximise everyone's work, but this is not in line with the main aim of the activity, which is to reach a solution by sharing ideas orally with written support on the board. In this activity, the aim of the written work is to enable the pupils to retain ideas or carry out calculations. (Agostino & de Versailles, 2024). The aim is for pupils to read and discuss the Instagram posts (M3) and comments:

- Who's right?

- Who's wrong?
- Why is this?

To defend their ideas, they must use the conjectures they have seen in previous lessons (e.g. "the assertion is false because we also saw a conjecture that seemed to work but no longer worked from n=40").

**Note:** Not all Instagram publications have the same level of difficulty. Here is a table summarising the publications and their difficulty.

| Publication | Features | Level of difficulty |
|---|---|---|
| 1 | • Abstract, with no specific theorem<br>• Positions clearly defined. The choice of winner remains to be made. | Easy |
| 2 | • Specific conjecture to understand<br>• One comment points out an erroneous line of reasoning | Medium |
| 3 | • Abstract statement<br>• Comment identifies the error and its source | Easy |
| 4 | • Large text<br>• Mentions the Riemann hypothesis without explanation (makes the task more complex)<br>• Ambiguity about who's right | Medium |
| 5 | • Explains twin primes (prerequisite for discussion)<br>• Extensive text<br>• Very abstract<br>• Complex reasoning about infinity | Difficult |
| 6 | • Abstract<br>• Complex reasoning about infinity | Difficult |

**Second phase (7 min):** At the end of the first phase, the groups change mini-classes. One pupil per group stays in their place (chosen by consensus or by the teacher) to present the Instagram publication studied previously to the new group, using the publication and the notes on the board as a basis. The newcomers take notes, ask questions or correct if necessary. This is also an opportunity to finish unfinished work. Note-taking is permitted because it encourages concentration and reproduces the usual attitude in class. This phase allows the learners to quickly reinvest what they have learned in the first stage.

Because of the change of groups, it is essential to distribute at least 2 different Instagram publications to the groups. You don't want a student who moves to find the same Instagram publication as the one they've already studied.

Phase 2 takes less time. The explanation by the remaining pupil is generally correct (the fruit of the previous collective reflection), and the essential part consists of understanding and answering the group's questions. This process is naturally quicker than the initial development of the solution.

**Closing (5 min)**: The session can end with a collective moment where each group briefly answers a question such as "What have you learnt today?" or "What have you gained from this session?". This period clearly marks a return to the collective, with the pupils expressing themselves for the first time in front of the whole class and the teacher. This activity enables the teacher to check that the pupils have acquired the vocabulary they need and to assess their ability to articulate logical reasoning. By speaking concisely on a subject they have now mastered, the pupils are placed in a more demanding context than that of the small working group.

# 04 Differentiation possibilities

As indicated in the previous section, there are two versions for all the conjectures: the blanc version (M1) and the guided version (M2). In the blanc version, students have to navigate through the conjectures without much help. Each problem is posed in the same way in the form of examples - conjecture - proof - theorem, and so the students receive a minimal amount of guidance, but are still left without help when faced with the different sub-sections. In the guided version, there are aids and gap-fill texts to guide the students.

Teachers are free to combine the two versions as they wish:

- blanc version for some students and guided version for others,
- blanc version for the beginning and guided version for the rest,
- blanc version for the first conjectures and guided version for the last conjectures,
- ...

# 05 Further criteria to be met as part of the lesson series

a. **Luxembourg context:** The University of Luxembourg has a very active mathematics department that is well represented on the international scene. Two of the department's researchers are featured in the interview in *1.6 A word from the scientists*.

b. **Differentiation:** As described in the previous paragraph, the module contains several levels of differentiation.

c. **Reference guide for media literacy[3] :**

- Competences 1: 1.2 Analysing and assessing data, information and digital content.
- Competences 2: 2.1 Working with others 2.3 Using appropriate forms of expression (netiquette).
- Competences 3: 3.3 Modelling, structuring and coding.

d. **4C model:** communication, collaboration, creativity, critical thinking: The 4Cs are used in this module. The various questions are solved by group work, which requires communication and collaboration on the part of the students.

e. **Link with mathematical research:** The activities in this module are much closer to mathematical research than the traditional mathematical exercises found in books. This module invites students to "be mathematicians for a day": to discover mathematical curiosities, formulate them in the form of conjectures and prove them.

[3] https://edumedia.lu/wp-content/uploads/2024/12/Medienkompass_EN_web.pdf

## Detailed planning of the lesson

| Duration | Phases | Focus | Social forms / Methods | Equipment / Materials | The learning process |
|---|---|---|---|---|---|
| Lesson one | | | | | |
| 15' | Getting started | Conjecture | Plenary session<br>Individual work | M1 & M2 | Students...<br>... know the definition of a conjecture.<br>... can read and interpret a mathematical statement.<br>... can produce several examples from a given example. |
| 30' | Work phase I | Examples and conjectures | Working in small groups | M1 & M2<br>Paper and/or tablets | Students...<br>... can produce several examples from a given example.<br>... know how to put together their individual examples.<br>... know how to formulate a mathematical conjecture based on examples. |
| Second hour of teaching | | | | | |
| 35' | Work phase II | Proof | Working in small groups | M1 & M2<br>Paper and/or tablets | Students...<br>... know how to reason about a mathematical problem.<br>... are making their first attempts at formulating evidence. |
| 10' | Conclusion | Various conjectures | Plenary session | M1 & M2 | Students...<br>... see different types of conjectures.<br>... value and understand the need for mathematical proof. |
| Third hour of teaching | | | | | |
| 10' | Correction | Correction of worksheets | Working individually or in small groups | M1 & M2 | Students...<br>... reread their work.<br>... evaluate their mistakes and the quality of their writing. |
| 10' | Getting started | Conjecture | Individual work | M1 & M2 | Students...<br>... know the definition of a conjecture.<br>... can read and interpret a mathematical statement.<br>... can produce several examples from a given example. |
| 25' | Work phase I | Examples and conjectures | Working in small groups | M1 & M2 | Students...<br>... can produce several examples from a given example. |

| | | | | | |
|---|---|---|---|---|---|
| | | | | Paper and/or tablets | ... know how to put together their individual examples.<br>... know how to formulate a mathematical conjecture based on examples. |
| Fourth hour of teaching | | | | | |
| 35' | Work phase II | Proof | Working in small groups | M1 & M2<br>Paper and/or tablets | Students...<br>... know how to reason about a mathematical problem.<br>... are making their first attempts at formulating evidence.<br>... discover the concept of a counterexample. |
| 10' | Conclusion | Various conjectures | Plenary session | M1 & M2 | Students...<br>... see different types of conjectures.<br>... value and understand the need for mathematical proof. |
| Fifth and sixth teaching hours | | | | | |
| (Plan identical to 3rd and 4th hour) | | | | | |
| Seventh (Fifth) teaching hour | | | | | |
| 10' | Correction | Correction of worksheets | Working individually or in small groups | M1 & M2 | Students...<br>... reread their work.<br>... evaluate their mistakes and the quality of their writing. |
| 5' | Setup | Setup | Setup of groups | Chairs benches | / |
| 18' | Phase I | Interpreting mathematical statements | Working in groups | M3 | Students...<br>... analyse mathematical statements.<br>... apply their knowledge from previous lessons to assess the truth of a statement.<br>... formulate a conclusion. |
| 7' | Phase II | Oral presentation | Working in groups | M3 | Students...<br>... present their findings.<br>... listen actively and ask questions.<br>... make connections between their work and that of other groups. |

| | | | | | |
|---|---|---|---|---|---|
| 5' | Closure | Mathematical vocabulary | Plenary session | M3 | Students...<br>... present their findings.<br>... review what they have learned. |

# 1.3 Teaching materials

# M1 Blanc version

# 01 | A strange calculation

## Introduction to problem

1. Choose a number.
2. Multiply it by 3.
3. Add 6.
4. Divide this result by 3.
5. Subtract the number chosen in step 1 from the answer in step 4.

## Exploration

Test the instructions on at least 3 examples. Can you give other examples?

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works, or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 02 | Happy Birthday!

## Introduction to problem

Take the number of the month of your birthday (1 for January, 2 for February, ...) and multiply it by 2. Add 5, then multiply the result by 50. Add the day of the month of your birthday. Subtract 250 to get a 4- or 3-digit number.

## Exploration

Analyse a few examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 03 | Multiply by 9

## Introduction to problem

Choose a number between 1 and 10. Multiply it by 9. Add the digits of the new number and add 4. What happens ?

## Exploration

Analyse a few examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 04 | Multiply by 6

## Introduction to problem

Take an even number and multiply it by 6. Compare the units digit of the result with the units digit of the number you started with. What do you find?

## Exploration

Analyse a few examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works, or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 05| Three figures become six figures

## Introduction to problem

1. Choose a three-digit number and write it twice to make a six-digit number. For example, 371371 or 552552.
2. Divide the number by 7.
3. Divide the result by 11.
4. Divide the result by 13.

## Exploration

Analyse some examples. You can use a spreadsheet or a calculator to do the calculations.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

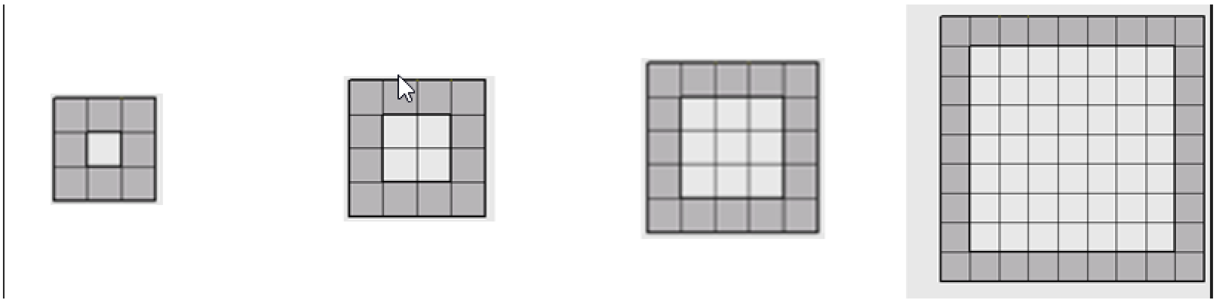
## Conclusion

Write a short commentary summarising your observations and thoughts.

# 06 | Mosaic tiles

## Introduction to problem

Here are some mosaic tiles consisting of white tiles forming squares of different sizes. Every white square is surrounded by a frame of grey tiles. Here are four examples.



Count the grey squares. What do you notice?

## Exploration

Analyse some examples. You can use a spreadsheet or a calculator to do the calculations.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 07 | The calendar

## Introduction to problem

The numbers for each month in the calendar can be grouped into squares of different sizes:

**January**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | | | | | | |

**February**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | | | | | |

**March**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 27 | 28 | 29 | 30 | 31 | | |

**April**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |

**May**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | | | | |

**June**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 26 | 27 | 28 | 29 | 30 | | |

Step 1: Draw a few squares on the calendar.

- Start by drawing squares of 2 by 2 days.
- Then try squares of 3 by 3 days.

Step 2: For each square you have drawn:

- Multiply the number in the top right-hand corner by the number in the bottom left-hand corner.
- Multiply the number on the top left with the number on the bottom right.
- Calculate the difference between these two products.

Here are two examples:

**February**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | | | | | |

$$11 \cdot 17 - 10 \cdot 18 = 7$$

**April**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |

$$14 \cdot 26 - 12 \cdot 28 = 28$$

What do you notice?

## Exploration

Analyse some examples. You can use a spreadsheet or a calculator to do the calculations.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule:

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 08 | Sum of odd numbers

## Introduction to problem

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

Do you spot a pattern?

## Exploration

Analyse more  examples:

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule:

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 09 | Add 41

## Introduction to problem

Choose a natural number $n$. Multiply it by itself. Add to the result the number you chose at the start, then add 41. Start with $n = 0$ , then $n = 1$ and so on. What do you find?

## Exploration

Analyse some examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 10 | 3's followed by a 1

## Introduction to problem

What are the divisors of 31? 331? And 3331? What do you find?

## Exploration

Analyse some examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 11 | Some rather unusual powers of 2

## Introduction to problem

Take a natural number $n$. Raise 2 to the power of $n$ to obtain the result $m$. Then raise 2 to the power $m$. Add 1 to the result.

Start with $n = 0$, then move on to $n = 1$ and so on. What do you notice?

## Exploration

Analyse some examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 12 | Finding common divisors

## Introduction to problem

Take a natural number $n$. Calculate the value of $A = n^2 + 7$ and $B = (n+1)^2 + 7$, then try to find common divisors of $A$ and $B$. Start with $n = 0$, then move on to $n = 1$ and so on. What do you find?

## Exploration

Analyse some examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## Any proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 13 | Even numbers and prime numbers

## Introduction to problem

$$4 = 2 + 2$$
$$6 = 3 + 3$$
$$8 = 5 + 3$$
$$10 = 7 + 3$$
$$12 = 7 + 5$$
$$14 = 7 + 7$$
$$16 = 11 + 5$$

Can you spot a pattern?

## Exploration

Aanysle some more examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 14 | An algorithm that always finishes?

## Introduction to problem

Choose a strictly positive integer.

- If it is even, divide by 2.
- If it is odd, multiply it by 3 and add 1 to the result.

Repeat this operation a large number of times to make the result as small as possible.

## Exploration

Analyse some examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule:

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 15 | A sum of 4 cubes

## Introduction to problem

$$1 = 1^3 + 0^3 + 0^3 + 0^3$$
$$2 = 1^3 + 1^3 + 0^3 + 0^3$$
$$3 = 1^3 + 1^3 + 1^3 + 0^3$$
$$4 = 1^3 + 1^3 + 1^3 + 1^3$$
$$5 = 2^3 + (-1)^3 + (-1)^3 + (-1)^3$$

What do you notice?

## Exploration

Analyse some examples.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

## A proof?

Try to explain why your rule always works, or look for counterexamples. You can use the technological tools at your disposal.

## Conclusion

Write a short commentary summarising your observations and thoughts.

# M2 Guided version

# 01 | A strange calculation

## Introduction to problem

1. Choose a number.
2. Multiply it by 3.
3. Add 6.
4. Divide this result by 3.
5. Subtract the number chosen in step 1 from the answer in step 4.

## Exploration

Analyse some examples by completing the table below. You can use a spreadsheet or a calculator to do the calculations.

| | Example 1 | Example 2 | Example 3 | Example 4 |
|---|---|---|---|---|
| Choose a number. | | | | |
| Multiply it by 3. | | | | |
| Add 6. | | | | |
| Divide by 3. | | | | |
| Subtract the number in the first line from the number in the 4$^{th}$line. | | | | |

## Conjecture

Describe the pattern observed.

If we carry out the algorithm described above, we always obtain

________________________________________.

## A proof?

Is this conjecture always true? We can **prove** it using the techniques of **algebra**.

Let's call $n$ the number chosen at the beginning.

Let's repeat the calculations made during the exploration (but using $n$):

| | |
|---|---|
| Choose a number. | $n$ |
| Multiply it by 3. | |
| Add 6. | |
| Divide by 3. | |
| Subtract the number in the first line from the number in the $4^{th}$ line. | |
| Simplify the expression. | |

The final result is ____, regardless of the number $n$ we start with.

## Conclusion

| | ✔ ? ✖ |
|---|---|
| If we run the following algorithm:<br>1. Choose a number.<br>2. Multiply it by 3.<br>3. Add 6.<br>4. Divide this result by 3.<br>5. Subtract the number at the beginning of step 1 from the answer in step 4.<br>the end result is always __________. | |

# 02 | Happy Birthday!

## Introduction to problem

Take the number of the month of your birthday (1 for January, 2 for February, ...) and multiply it by 2 . Add 5 , then multiply the result by 50 . Add the day of the month of your birthday. Subtract 250 to get a 4- or 3-digit number.

## Exploration

Analyse some examples by completing the table below. You can use a spreadsheet or a calculator to do the calculations.

| | Student 1 | Student 2 | Student 3 | Student 4 |
|---|---|---|---|---|
| Take the month of your birthday. | | | | |
| Multiply by 2. | | | | |
| Add 5. | | | | |
| Multiply by 50. | | | | |
| Add the day of the birthday. | | | | |
| Subtract 250. | | | | |

Compare the final results with the birthdays that you started with.

## Conjecture

Complete the following text to describe the phenomenon observed:

The final result shows _______________ of a birthday as follows:

6. The number formed by the _______________ digit and the _______________ digit of the final result represents the month of the birthday.
7. The number formed by the ______________ digit and the _________ digits of the final result represents the _______________ of the birthday.

## A proof?

Does this rule always work? There are two ways to find out:

Option 1:

Try **all** possible **dates**: from 01.01 to 31.12. How many cases do you need to check?

To assess such a large number of different cases, it is advisable to use a **spreadsheet** or a **programming language** to automate the calculations.

After assessing all the cases, we found that

Option 2:

We'll try to **prove** the conjecture using the techniques of **algebra**.

Let's call the day of a person's birthday $d$.

Let's call a person's month $m$.

| | |
|---|---|
| Take the month of the birthday. | $m$ |
| Multiply by 2. | |
| Add 5. | |
| Multiply by 50. | |
| Add the day $d$ of the birthday. | |
| Subtract 250. | |
| Simplify the expression obtained. | |

The final result is _______________ .

The last two digits of this result represent ______________________________________.

The first digit or first two digits of this result represent ________________________.

## Conclusion

| | |
|---|---|
| If we follow the following algorithm :<br><br>Take the number of the month of your birthday.<br>Multiply by 2.<br>Add 5.<br>Multiply by 50.<br>Add the day of the month of your birthday.<br>Substract 250.<br><br>this gives :<br><br>▪ a three-digit number if the month of the birthday is less than or equal to _____. The hundreds digit of this number represents _________ and the last two digits represent the _________ of the birthday.<br><br>▪ a four-digit number if the month of the birthday is greater than or equal to _____. The first two digits of this number represent _________ and the last two digits represent the _________ of the birthday. | |

# 03 | Multiply by 9

## Introduction to problem

Choose a number between 1 and 10. Multiply it by 9. Add the digits of the new number and add 4. What happens?

## Exploration

Analyse a few examples.

| | Example 1 | Example 2 | Example 3 | Example 4 | Example 5 |
|---|---|---|---|---|---|
| Choose a number between 1 and 10. | 1 | 2 | 3 | 4 | 5 |
| Multiply by 9. | | | | | |
| Add up the digits. | | | | | |
| Add 4. | | | | | |

We can see that the result of this algorithm always seems to be equal to ___________.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

If we take a number between 1 and 10, multiply it by 9, then calculate the sum of the digits in the result and finally add 4, we obtain the number _______.

## A proof?

In fact, we only have _____ other cases to check. If we manage to check our rule for these cases, our conjecture is proven:

| | | | | | |
|---|---|---|---|---|---|
| Choose a number between 1 and 10. | | | | | |
| Multiply it by 9. | | | | | |
| Add up the digits. | | | | | |
| Add 4. | | | | | |

Does the rule also work for numbers greater than 10? Try some examples:

| | | | | | |
|---|---|---|---|---|---|
| Choose a number greater than 10 | | | | | |
| Multiply it by 9. | | | | | |
| Add up the digits. | | | | | |
| Add 4. | | | | | |

What can you conclude?

| | | | |
|---|---|---|---|
| The rule | ☐<br>☐ | works<br>does not work | for numbers greater than 10. |

## Conclusion

| | ✓ ? ✗ |
|---|---|
| If we take any integer between 1 and 10 and follow the following algorithm:<br>Multiply it by 9.<br>Add up the digits.<br>Add 4.<br>the end result is always _________. | |

# 04 | Multiply by 6

## Introduction to problem

Take an even number and multiply it by 6. Compare the units digit of the result with the units digit of the number you started with. What do you find?

## Exploration

Analyse a few examples.

| | Example 1 | Example 2 | Example 3 | Example 4 |
|---|---|---|---|---|
| Choose an even number. | 4 | 12 | | |
| Multiply it by 6. | | | | |
| Units digit of the result. | | | | |
| Units digit of the number chosen at the start. | | | | |

It seems that the units digit of the result of this algorithm is always equal to ______________________________________________.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule:

If an even number is multiplied by 6, then the units digit of the result equals ______________________________________________.

## A proof?

Let's call the number at the beginning $n$.

| | |
|---|---|
| Since n is even, $n$ can be written as... | |
| Multiply it by 6. | |
| Simplify the expression. | |

The final result is _____________.

Write this result as the sum of two terms:

_________ = __________ + _________

The units digit is shown here.

The units digit equals ______________________________________________________.

## Conclusion

We have proved the following theorem:

| | ✓ ? ✗ |
|---|---|
| If an even number is multiplied by 6, then the units digit of the result equals<br>_________________________________________________. | |

# 05| Three digits become six digits

## Introduction to problem

1. Choose a three-digit number and write it twice to make a six-digit number. For example, 371371 or 552552.
2. Divide the number by 7.
3. Divide the result by 11.
4. Divide the result by 13.

## Exploration

Analyse some examples by completing the table below. You can use a spreadsheet or a calculator to do the calculations.

| | Example 1 | Example 2 | Example 3 | Example 4 |
|---|---|---|---|---|
| Choose a three-digit number | | | | |
| Write it down twice to get a 6-digit number. | | | | |
| Divide by 7. | | | | |
| Divide by 11. | | | | |
| Divide by 13. | | | | |

## Conjecture

Describe the phenomenon observed :

If we choose a 3-digit number and run the algorithm described above, we get ______________________________________________.

## A proof?

Does this rule still work? To check, we can use the techniques of **algebra**.

Let's call the 3-digit number at the beginning $n$.

Step 1:

Let's construct the 6-digit number obtained by repeating the number $n$ twice.

| |
|---|
| Hint: Use your calculator. Type in the 3-digit number $n$. What calculation can you do so that the calculator displays the 6-digit number made up of twice the 3-digit number? |

The 6-digit number is written as follows (using $n$): __________________________

Step 2:

Let's continue running the algorithm.

| | |
|---|---|
| Choose a three-digit number. | $n$ |
| Write it down twice to get a 6-digit number. | |
| Divide by 7. | |
| Divide by 11. | |
| Divide by 13. | |
| Write the expression as a fraction. | |
| Simplify the fraction as far as possible. | |

The resulting number is : ______________________________________________

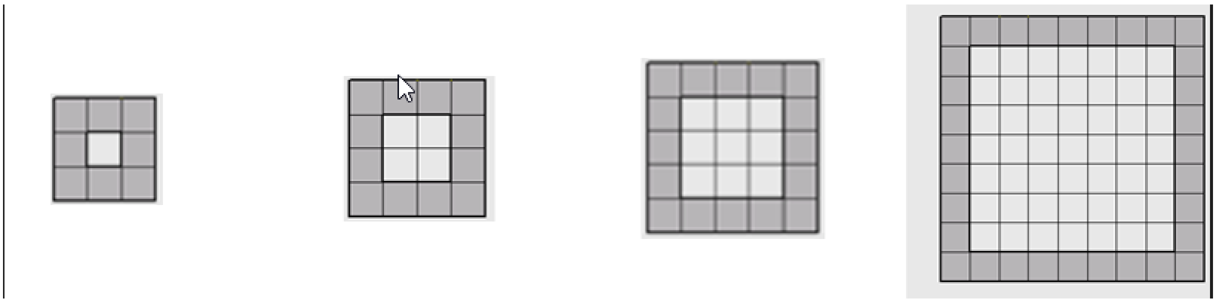
## Conclusion

| | ✓ ? ✗ |
|---|---|
| We choose a three-digit number and apply the following algorithm:<br>1. Write it down twice to get a six-digit number.<br>2. Divide the number by 7.<br>3. Divide the result by 11.<br>4. Divide the result by 13.<br>Then we always get ________________________________________ . | |

# 06 | Mosaic tiles

## Introduction to problem

Here are some mosaic tiles consisting of white tiles forming squares of different sizes. Every white square is surrounded by a frame of grey tiles. Here are four examples.



Count the grey squares. What do you notice?

## Exploration

Analyse some examples by completing the table below.

| Number of white tiles along one edge | Drawing | Number of grey tiles |
|---|---|---|
| 4 | | |
| 5 | | |
| 6 | | |
| | | |

| | | |
|---|---|---|
| | | |
| | | |

## Conjecture

Describe the phenomenon observed :

The number of grey squares surrounding a square of $n$ by $n$ white squares is equal to ________.

## A proof?

Does this rule still work? To check, we can use the techniques of **algebra**.

Let's call $n$ the size of the white square.

| | |
|---|---|
| Number of white tiles along one edge of the square. | $n$ |
| Number of tiles along one edge in the white-and-grey square. | |
| The number of grey tiles. | |
| Simplify. | |

If the white square is of size $n \times n$, the number of grey squares is _________________________.

## Conclusion

| | |
|---|---|
| The number of grey squares surrounding a square with $n$ white tiles along one edge is equal to ________. | |

# 07 | The calendar

## Introduction to problem

The numbers for each month in the calendar can be grouped into squares of different sizes:

**January**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | | | | | | |

**February**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | | | | | |

**March**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 27 | 28 | 29 | 30 | 31 | | |

**April**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |

**May**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | | | | |

**June**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 26 | 27 | 28 | 29 | 30 | | |

Step 1: Draw a few squares on the calendar.

- Start by drawing squares of 2 by 2 days.
- Then try squares of 3 by 3 days.

Step 2: For each square you have drawn:

- Multiply the number in the top right-hand corner by the number in the bottom left-hand corner.
- Multiply the number on the top left with the number on the bottom right.
- Calculate the difference between these two products.

Here are two examples:

**February**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | | | | | |

$$11 \cdot 17 - 10 \cdot 18 = 7$$

**April**

| S | M | T | W | T | F | S |
|---|---|---|---|---|---|---|
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |

$$14 \cdot 26 - 12 \cdot 28 = 28$$

What do you notice?

## Exploration

Draw squares of different sizes in the following calendar. Then complete the table.

| janvier | | | | | | 1 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | | | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 25 | 26 | 27 | 28 | 29 | 30 | 31 |
| | | | | | | |

| février | | | | | | 2 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| | | | | | | |
| | | | | | | |

| mars | | | | | | 3 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | | | | |
| | | | | | | |

| avril | | | | | | 4 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 26 | 27 | 28 | 29 | 30 | | |
| | | | | | | |

| mai | | | | | | 5 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | | | | | | |

| juin | | | | | | 6 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | 30 | | | | |
| | | | | | | |

| juillet | | | | | | 7 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 26 | 27 | 28 | 29 | 30 | 31 | |
| | | | | | | |

| août | | | | | | 8 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | | | | | 1 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 30 | 31 | | | | | |

| septembre | | | | | | 9 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 27 | 28 | 29 | 30 | | | |
| | | | | | | |

| octobre | | | | | | 10 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | | | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 25 | 26 | 27 | 28 | 29 | 30 | 31 |

| novembre | | | | | | 11 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | | | | | |

| décembre | | | | | | 12 |
|---|---|---|---|---|---|---|
| L | M | M | J | V | S | D |
| | | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 27 | 28 | 29 | 30 | 31 | | |

| Square size | Number top right corner | Number bottom left corner | Product of the two | Number top left corner | Number bottom right corner | Product of the two | Difference |
|---|---|---|---|---|---|---|---|
| 2 | | | | | | | |
| 2 | | | | | | | |
| 2 | | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | | | | | | | |
| 3 | | | | | | | |
| 3 | | | | | | | |
| 4 | | | | | | | |
| 4 | | | | | | | |
| 4 | | | | | | | |

## Conjecture

Describe the phenomenon observed :

We see that the difference between the product of the number in the top right-hand corner with the number in the bottom left-hand corner and the product of the number in the top left-hand corner with the number in the bottom right-hand corner of a square of $n$ by $n$ days is equal to

_________________.

## A proof?

Does this rule still work? To check, we can use the techniques of **algebra**.

Let's call $n$ the size of the square and let's call $a$ the number in the top left-hand corner.

| | |
|---|---|
| Top left corner | $a$ |
| Top right corner | |
| Bottom left corner | |
| Bottom right corner | |
| Product of the number in the top right corner with the number in the bottom left corner | |
| Product of the number in the top left corner and the number in the bottom right corner | |
| Difference between the two products | |
| Simplify the expression. | |

## Conclusion

| The difference between the product of the number in the top right corner with the number in the bottom left corner and the product of the number in the top left corner with the number in the bottom right corner of a square of size $n$ is equal to _______________. | ✓ ? ✗ |
|---|---|

# 08 | Sum of odd numbers

## Introduction to problem

$$
\begin{aligned}
1 &= 1\\
1+3 &= 4\\
1+3+5 &= 9\\
1+3+5+7 &= 16\\
1+3+5+7+9 &= 25\\
&\dots\\
1+3+5+\cdots+197+199 &= 100\,000
\end{aligned}
$$

Do you spot a pattern?

## Exploration

Use a spreadsheet or calculator to complete the following table:

| Step $k$ | | |
|---|---|---|
| 1 | $1 =$ | $=$ |
| 2 | $1+3 =$ | $=$ |
| 3 | $1+3+5 =$ | $=$ |
| 4 | $1+3+5+7 =$ | $=$ |
| 5 | $1+3+5+7+9 =$ | $=$ |
| 6 | $1+3+5+7+9+11 =$ | $=$ |
| 7 | $1+3+5+7+9+11+13 =$ | $=$ |

## Conjecture

Based on the results of the exploration above, try to formulate a general rule:

The sum of $k$ odd numbers is ____________________ and is equal to

_________.

## A proof?

We are going to prove the conjecture for sums up to 100. There are_________ cases to check. As this is a relatively large number, it is reasonable to use a **spreadsheet** or a **programming language**:

In this case, we suggest you use a spreadsheet program (such as Excel, Numbers, Google Sheets, etc.) and draw up a list similar to the one below:

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1–2 | Step k | k-th odd number | sum of the first k odd numbers | Conjecture | Check |
| 3 | 1 | 1 | 1 | 1 | TRUE |
| 4 | 2 | 3 | 4 | 4 | TRUE |
| 5 | 3 | 5 | 9 | 9 | TRUE |
| 6 | 4 | 7 | 16 | 16 | TRUE |
| 7 | 5 | 9 | 25 | 25 | TRUE |
| 8 | 6 | 11 | 36 | 36 | TRUE |
| 9 | 7 | 13 | 49 | 49 | TRUE |
| 10 | 8 | 15 | 64 | 64 | TRUE |
| 11 | 9 | 17 | 81 | 81 | TRUE |
| 12 | 10 | 19 | 100 | 100 | TRUE |
| 13 | 11 | 21 | 121 | 121 | TRUE |
| 14 | 12 | 23 | 144 | 144 | TRUE |
| 15 | 13 | 25 | 169 | 169 | TRUE |

To check all the possible cases, you need to work with **formulas**:

In column A, start by writing 1 in box A3. Then in box A4, write a formula that adds 1 to the box above (A3+1). Then drag this formula down to fill the whole column. The result is: 1, 2, 3, 4...

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1–2 | Step k | k-th odd number | sum of the first k odd numbers | Conjecture | Check |
| 3 | 1 | | | | |
| 4 | =A3+1 | | | | |
| 5 | | | | | |

In column B, we also write 1 in box B3. But in box B4, we add 2 to the box above (B3+2). Pull this formula downwards. You get: 1, 3, 5, 7...

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1-2 | Step k | k-th odd number | sum of the first k odd numbers | Conjecture | Check |
| 3 | 1 | 1 | | | |
| 4 | 2 | =B3+2 | | | |
| 5 | 3 | | | | |

For column C, in box C4, add the value of C3 to that of B4. Then pull this formula down. This column allows us to see how the numbers add up.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1-2 | Step k | k-th odd number | sum of the first k odd numbers | Conjecture | Check |
| 3 | 1 | 1 | 1 | | |
| 4 | 2 | 3 | =C3+B4 | | |
| 5 | 3 | 5 | | | |

In column D, we're going to test our idea: we write our formula in box D3 and pull it down.

Column E will be used to check whether our idea is correct or not.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1-2 | Step k | k-th odd number | sum of the first k odd numbers | Conjecture | Check |
| 3 | 1 | 1 | 1 | 1 | =C3=D3 |
| 4 | 2 | 3 | 4 | 4 | |

This makes it easy to check all the cases for $k$ from1 to _________.

## Conclusion

| | |
|---|---|
| The sum of $k$ odd numbers is ________ and is equal to ________ (for $k$ from1 to _________ ). | |

**Does the conjecture hold true for values of $k$ greater than $100$ ?**

Search the Internet or ask artificial intelligence for help in finding answers!

# 09 | Add 41

## Introduction to problem

Choose a natural number $n$ . Multiply it by itself. Add to the result the number you chose at the start, then add 41. Start with $n = 0$ , then $n = 1$ and so on. What do you find?

## Exploration

Analyse some examples.

| $n$ | Multiply $n$ by itself | Add $n$ | Add 41 | Final result | Divisors of the final result |
|---|---|---|---|---|---|
| 0 | | | | | |
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

The final results are all ____________________________numbers.

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

If you multiply a number by itself, then add the number you chose at the start and finally add 41, you get ____________________________.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

Check your conjecture up to n=40 and n=41. Use a spreadsheet (or do it by hand if you like). What do you find?

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 10 | 3s followed by a 1

## Introduction to problem

What are the divisors of 31? 331? And 3331? What do you find?

## Exploration

Analyse a few examples.

| Number | Dividers |
|---|---|
| 31 | {1, 31} |
| 331 | |
| 3331 | |
| | |
| | |
| | |
| | |

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule:

All numbers of the form 333...1 are _________________.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

Check your guess up to step 8. Use a spreadsheet (or do it by hand if you like). What do you find?

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 11 | Some rather unusual powers of 2

## Introduction to problem

Take a natural number $n$ . Raise 2 to the power of $n$ to obtain the result $m$. Then raise 2 to the power $m$. Add 1 to the result.

Start with $n = 0$, then move on to $n = 1$ and so on. What do you notice?

## Exploration

Analyse some examples:

| $n$ | $2^n$ | $2^{2^n}$ | $2^{2^n} + 1$ |
|---|---|---|---|
| 0 | $2^0 = 1$ | $2^1 = 2$ | $2 + 1 = 3$ |
| 1 | $2^1 = 2$ | $2^2 = 4$ | $4 + 1 = 5$ |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule:

All numbers of the form $2^{2^n} + 1$ are ____________________________.

?

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

Check your conjecture up to $n = 5$ . Use a calculator or spreadsheet to do the calculations. Use artificial intelligence or a search engine to check your guess. What do you find?

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 12 | Finding common divisors

## Introduction to problem

Take a natural number $n$. Calculate the value of $A = n^2 + 7$ and $B = (n+1)^2 + 7$, then try to find common divisors of $A$ and $B$. Start with $n = 0$, then move on to $n = 1$ and so on. What do you find?

## Exploration

Analyse some examples.

| $n$ | $A = n^2 + 7$ | $B = (n+1)^2 + 7$ | gcd(A,B) |
|---|---|---|---|
| 0 | 7 | 8 | 1 |
| 1 | 8 | 11 | 1 |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

All numbers of the form $n^2 + 7$ and $(n+1)^2 + 7$ are

________________________________________.

## Any proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

Check your conjecture up to $n = 14$ . Use a calculator or spreadsheet to do the calculations. What do you find?

## Conclusion

Write a short commentary summarising your observations and thoughts.

# 13 | Even numbers and prime numbers

## Introduction to problem

$$
\begin{aligned}
4 &= 2 + 2 \\
6 &= 3 + 3 \\
8 &= 5 + 3 \\
10 &= 7 + 3 \\
12 &= 7 + 5 \\
14 &= 7 + 7 \\
16 &= 11 + 5 \\
&\dots \\
100 &= 41 + 59
\end{aligned}
$$

Can you spot a pattern?

## Exploration

Here are a few more examples:

| | |
|---|---|
| $16 = 11 + 5$ | 11 and 5 are ________________ numbers. |
| $18 =$ | ________ and ________ are ________________ numbers. |
| $20 =$ | ________ and ________ are ________________ numbers. |
| $22 =$ | ________ and ________ are ________________ numbers. |
| $24 =$ | ________ and ________ are ________________ numbers. |
| $26 =$ | ________ and ________ are ________________ numbers. |
| $28 =$ | ________ and ________ are ________________ numbers. |
| $30 =$ | ________ and ________ are ________________ numbers. |

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

Any ______________ number less than or equal to 100 can be written as

the sum of two ______________ numbers.

## Any proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

Search the Internet or ask artificial intelligence for help in finding answers!

You can also use the following script to find solutions:

main.py › ...

```python
import sympy

a=int(input('Enter an even number:'))

for p in sympy.primerange(0, 200):
  if sympy.isprime(a-p):
    print(a,'=',p,'+',a-p)

```

Can you rewrite any even number less than or equal to 100 as the sum of two primes?
☐ Yes ☐ No

A counterexample?

________________________________________________________ .

Are there any even numbers greater than 100 that cannot be rewritten as the sum of two primes?

________________________________________________________ .

## Conclusion

| | |
|---|---|
| Any _______________ number less than or equal to100 can be written as the sum of two______________numbers. | |
| Any _______________ number can be written as the sum of two _______________ numbers.<br>This problem is known as the __________________________. | |

# 14 | An algorithm that always finishes?

## Introduction to problem

Choose a strictly positive integer.

- If it is even, divide by 2.
- If it is odd, multiply it by 3 and add 1 to the result.

Repeat this operation a large number of times to make the result as small as possible.

## Exploration

Analyse some examples.

| Natural number between 1 and 100 | Intermediate results | Smallest result |
|---|---|---|
| 1 | 4; 2; 1; 4; 2; 1; … | 1 |
| 2 | 1; 4; 2; 1; 4; 2; 1; … | 1 |
| 3 | 10; 5; 16; 8; 4; 2; 1; 4; 2; 1; … | 1 |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

For integers chosen between 1 and 100, the algorithm described above always results in the value _______ after a finite number of iterations.

## A proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

Search the Internet or ask artificial intelligence for help in finding answers!

You can also test the conjecture on several cases on the following website: https://www.dcode.fr/conjecture-syracuse

For all numbers between 1 and 100 , does the algorithm always arrive at the number 1 after a finite number of iterations? ☐ Yes ☐ No

Is there a counterexample?

_______________________________________________________ .

What if you take starting numbers greater than 100?

_________________________________________________________________________

_________________________________________________________________________

## Conclusion

| | ✔ ? ✖ |
|---|---|
| For integers chosen between 1 and 100, the algorithm described above always results in the value _______ after a finite number of iterations. | |
| For **any positive integer**, the algorithm described above always results in the value _______ after a finite number of iterations.<br>This problem is known as the ___________________________. | |

# 15 | A sum of 4 cubes

## Introduction to problem

$$\begin{aligned}
1 &= 1^3 + 0^3 + 0^3 + 0^3 \\
2 &= 1^3 + 1^3 + 0^3 + 0^3 \\
3 &= 1^3 + 1^3 + 1^3 + 0^3 \\
4 &= 1^3 + 1^3 + 1^3 + 1^3 \\
5 &= 2^3 + (-1)^3 + (-1)^3 + (-1)^3 \\
&\dots \\
13 &= 10^3 + 7^3 + 1^3 + (-11)^3
\end{aligned}$$

What do you notice?

## Exploration

Analyse some examples.

| | | |
|---|---|---|
| $6 =$ | $8 - 1 - 1 + 0$ | $=$ |
| $7 =$ | $8 - 1 + 0 + 0$ | $=$ |
| $8 =$ | | $=$ |
| $9 =$ | | |

## Conjecture

Based on the results of the previous exploration, try to formulate a general rule.

Any integer between 1 and 13 can be written as

__________________________________________________________________.

## Any proof?

Try to explain why your rule always works or look for counterexamples. You can use the technological tools at your disposal.

| | | |
|---|---|---|
| Do some research on the Internet or ask artificial intelligence for help to check the remaining cases! | | |
| $10 =$ | | $=$ |
| $11 =$ | | $=$ |
| $12 =$ | | $=$ |

For all numbers between 1 and 13, can they be rewritten as ______________________________________ ? ☐ Yes ☐ No

Is there a counterexample?

______________________________________________________________ .

What if you use numbers greater than 13?

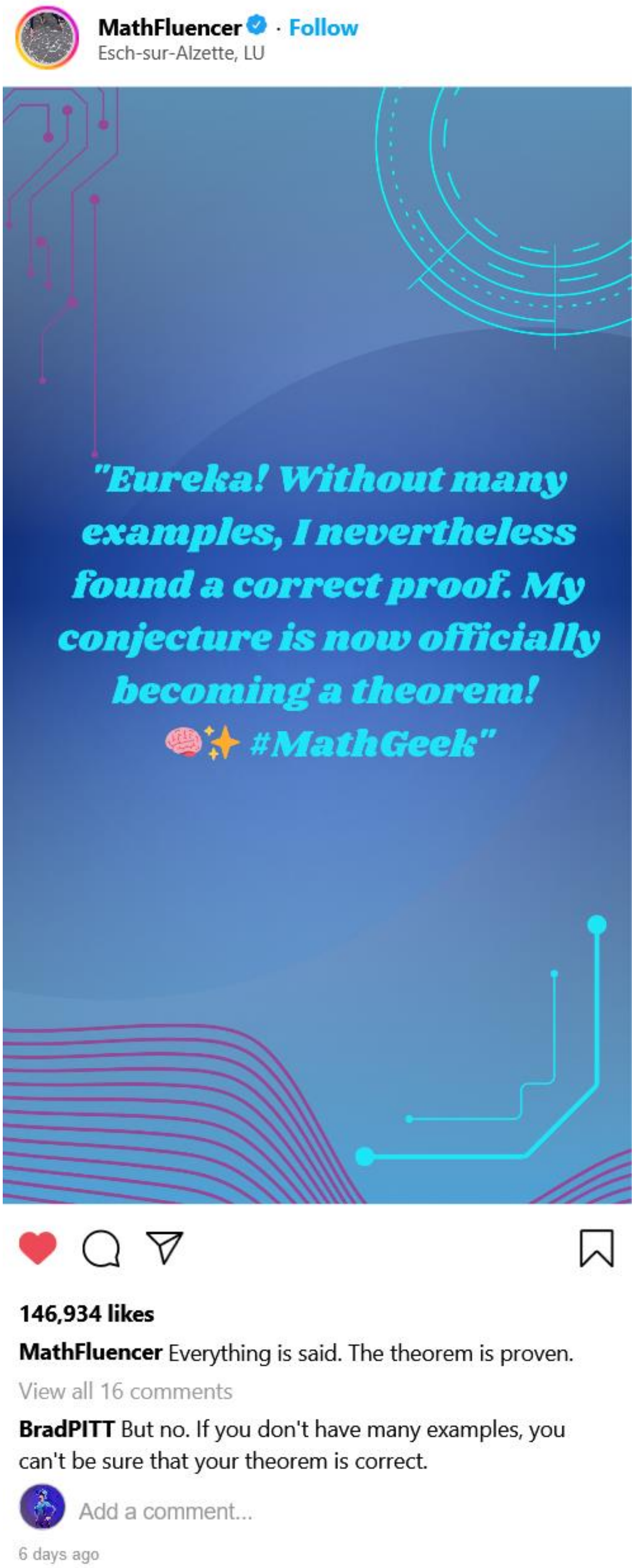
__________________________________________________________________________________

__________________________________________________________________________________

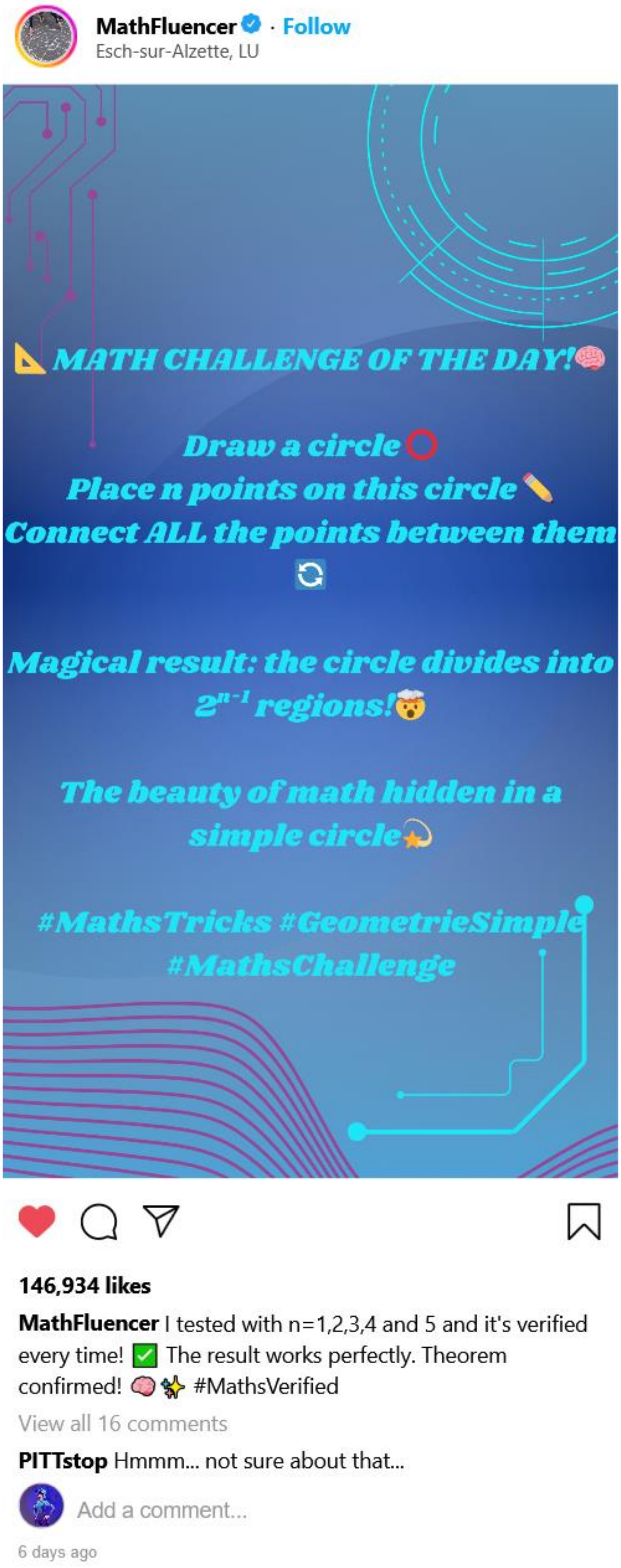
## Conclusion

| | |
|---|---|
| Any integer between 1 and 13 can be written as ___________________________________________________________________. | |
| Any **positive integer** can be written as __________________________________________________________________. This problem is known as the ___________________________________. | |

# M3 Instagram publications




Instagram post 1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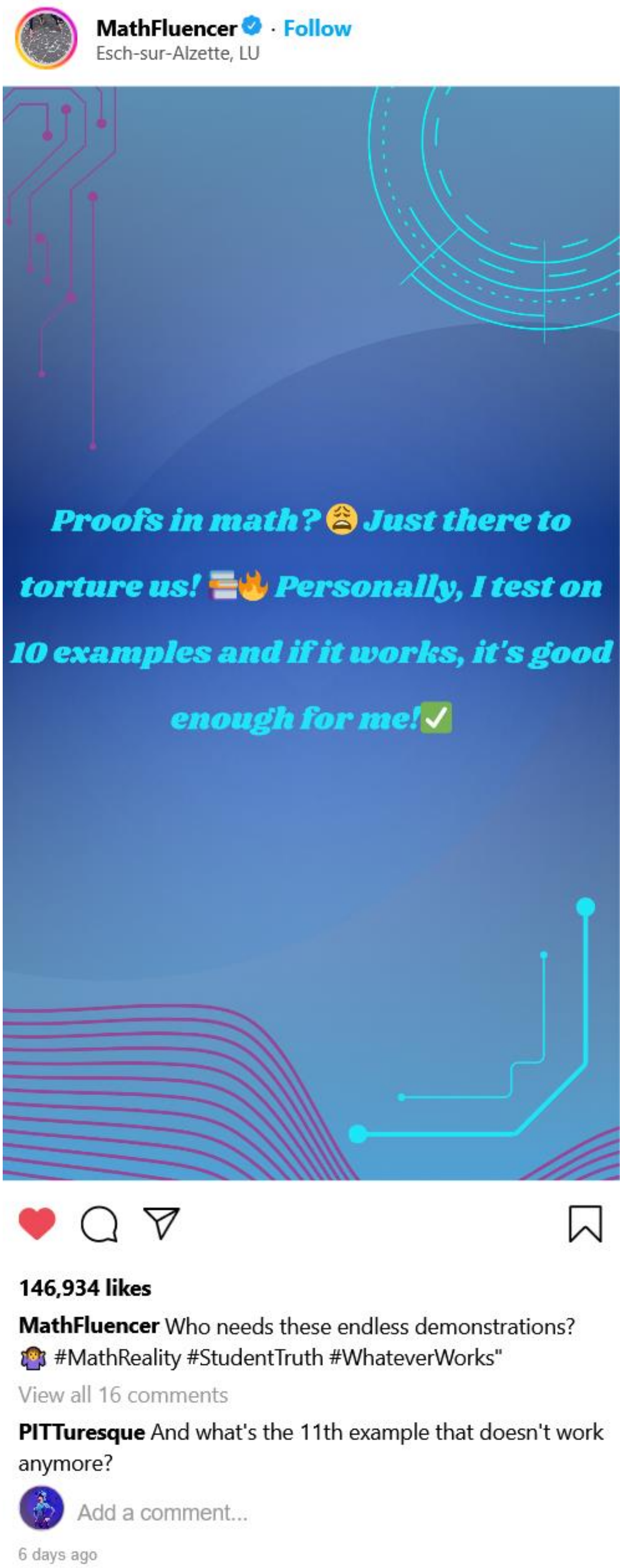
Esch-sur-Alzette, LU
📐 MATH CHALLENGE OF THE DAY! 🧠
Draw a circle ⭕
Place n points on this circle ✏️
Connect ALL the points between them 🔄
Magical result: the circle divides into $2^{n-1}$ regions! 🤯
The beauty of math hidden in a simple circle 💫
#MathsTricks #GeometrieSimple #MathsChallenge
146,934 likes
MathFluencer I tested with n=1,2,3,4 and 5 and it's verified every time! ✅ The result works perfectly. Theorem confirmed! 🧠✨ #MathsVerified
View all 16 comments
PITTstop Hmmm... not sure about that...
Add a comment...
6 days ago

Instagram publication 2



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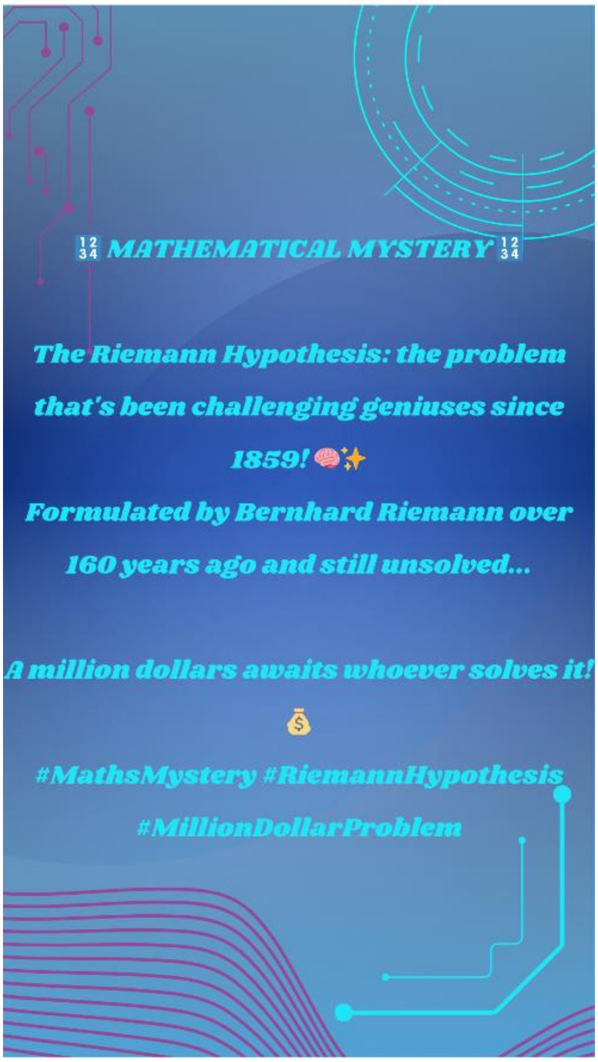


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**MathFluencer** The Holy Grail of mathematics remains unconquered. Are you the next to try your luck? 😉

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**PITTsOfWisdom** Stop 😂 No matter what... If nobody found a proof in the last 160 years, it's fake. 🤡 #FactsOnly #RealityCheck #CriticalThinking

**MathFluencer** It's not necessarily 😤 To be sure this conjecture is false, you'd need to find a counter-example, aka an example that doesn't work 📚 #MathLogic #NecessaryProofs #NoHastyConclusions

**PITTsOfWisdom** Maths is so boring 😴 #BoringAF #WhateverWorks #LiveTheRecreation#NecessaryProofs #NoHastyConclusions

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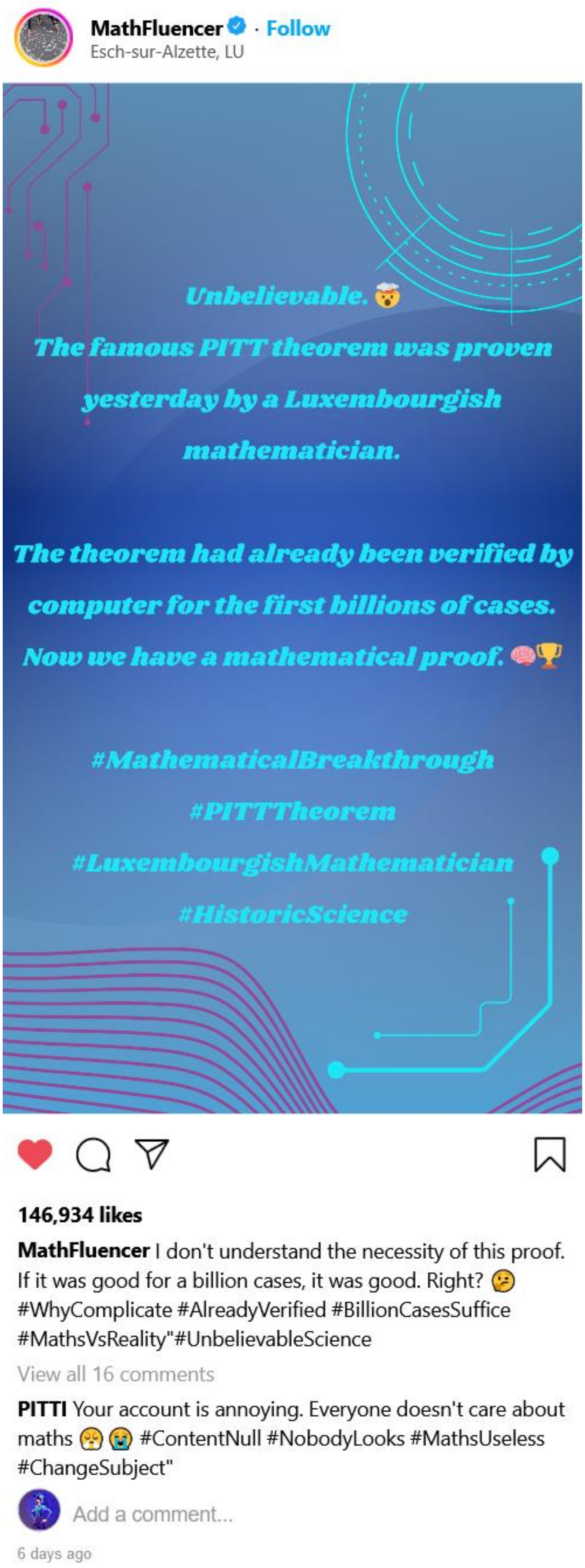
Instagram post 4


Instagram Post 5




Instagram Post 6

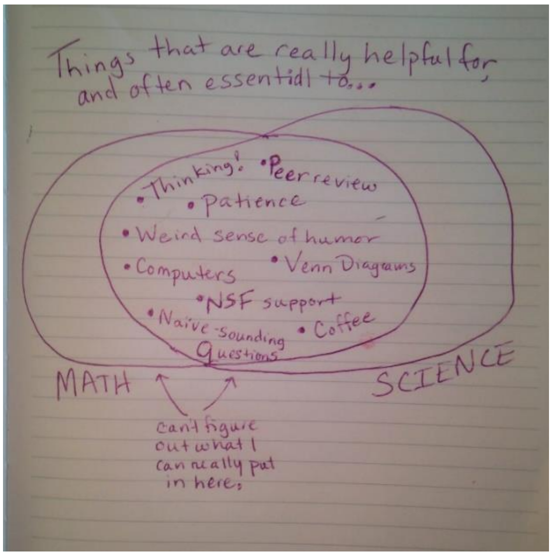
# 1.4 Interdisciplinary ideas

## Sciences

One of the primary aims of the sciences course is to impart a scientific approach. This involves a scientific way of thinking when dealing with problems. Pupils must learn the experimental approach, which places them in the position of a scientific researcher and enables them to construct their knowledge independently. The scientific approach involves gathering, structuring and interpreting information.[4] More specifically, the following skills are targeted:

1. Formulate well-founded hypotheses based on the context, using examples.
2. Make targeted observations, present and evaluate the results, and draw conclusions.
3. Recognise simple cause-and-effect relationships and draw logical conclusions.
4. Diagrammatic representation and interpretation of the models presented. [5]

The mathematical approach is not the same as that used in the natural sciences. Scientific theories are supported by repeated experiments, but are never proved, whereas we can actually prove things in mathematics. However, there is also a great similarity between the two. Mathematician Joseph Silverman explains in (Silverman, 2012) that in number theory, we collect data by calculating a large number of examples. Then we look for a model, make a hypothesis and test it with more data (more examples). If the hypothesis does not match the new data, we revise it. After a few iterations, when the new data matches our hypothesis, we finally try to prove it. This follows exactly the same pattern as in the natural sciences.




Source : Math is Like Science, Only with Proof. Math. AMS Blogs. https://blogs.ams.org/phdplus/2017/04/17/math-is-like-science-only-proof-y/

It is this aspect of the natural sciences that is emphasised in this module. Pupils learn to carry out experiments (in the mathematical context, this means doing several examples), to recognise patterns and models in order to formulate a hypothesis and finally to prove or disprove it. In addition, the use of algebraic tools contributes to the implementation of the scientific approach, complementing observation, manipulation and experimentation (Éduscol, 2016).

[4] Programme de Sciences Naturelles 7C and 6C.
[5] idem

## Philosophy, moral education

This module shows students how to demonstrate a fact in mathematics. The idea is that, in mathematics, a conjecture is true if and only if it is proved. Even if it has been verified on a billion examples, it remains a conjecture and we cannot be sure that it is true. This idea fits in with the following more general and philosophical question:

What does it mean to say that something is true?

In a slightly more restrictive setting, we can ask when something is true in science. This last idea is present in the "Vie et Société" syllabus for secondary 6 in the "Big Questions" section under the theme "Religion and Natural Sciences - What can I know?[6]

| *Themenfelder* | *Eigen-verantwortung* | *Anerkennung und Ausgrenzung* | *Demokratie als Teilhabe und Mitgestaltung* | *Prozess und Auswirkungen von Globalisierung* | *Zeichen und Symbole im Alltag* | *Glauben Wissen Meinen* |
|---|---|---|---|---|---|---|
| **8 Kl.** | Gesundheit, Sucht und Risiko-verhalten<br>------------<br>Liebe ist…? | Geschlechter-rollen<br>------------<br>Jung und Alt: Generationen<br>------------<br>Beeinträchtigung / Behinderung<br>------------<br>Stereotypen und Vorurteile | Integration und Partizipation in Schule und Gesellschaft<br>------------<br>Begegnungen mit versch. Religionen: Christentum | Armut, Wohl-fahrt, Reichtum<br>------------<br>Globalisierung<br>------------<br>Nachhaltigkeit und Fairer Handel<br>------------<br>Globale Um-welt-politik und lokales Leben | Ausdrucksformen in religiösen Gruppen und Gemeinschaften<br>------------<br>Kleidung und Mode als Ausdrucks-möglichkeit | Religion und Naturwissen-schaft<br>------------<br>Was kann ich wissen? – Was darf ich hoffen? |

It is therefore possible to discuss these questions during the course "Vie et Société" or any other philosophical course, at the same time as teaching this module during the mathematics course.

The Lumni philosophy sheet (in French) below discusses the essential distinction between example and proof, a fundamental conceptual nuance. This reflection fits perfectly into this module, which mathematically demonstrates the differences between three key epistemological concepts: proof (rigorous and universal demonstration), example (particular illustrative case) and counterexample (particular case that invalidates a general proposition).

Lumni
Exemple / Preuve

https://www.lumni.fr/article/exemple-preuve

[6] Rahmenlehrplan für das Fach Leben und Gesellschaft / Life and Society. https://vieso.script.lu/sites/default/files/2020-12/Rahmenlehrplan%20VIESO1_0.pdf

## Thinking about mathematics: a philosophical approach

The web documentary "Paroles de déchiffreurs" features fourteen interviews with researchers working in different areas of mathematics (more information under *1.5 More on this Topic*).

The content of this lesson lends itself to a rich interdisciplinary approach. The web page associated with the web documentary contains videos of the interviews filmed, as well as the texts of the interviews. The class will explore these varied texts, ranging from biographies of mathematicians to interviews with contemporary researchers, providing a basis for discussion that transcends the traditional boundaries between disciplines. These resources will make it possible to lead stimulating debates touching on mathematics as well as philosophy, physics, astronomy and even the history of science and the arts. This multi-dimensional approach will help to understand how mathematics fits into the wider context of human thought and culture.

*References*

1. Éduscol. (2016). Utiliser le calcul littéral. https://eduscol.education.fr/document/17263/download
2. Silverman, J. (2012). *A Friendly Introduction to Number Theory. Chapter 1.* Fourth Edition. Pearson Education, Inc. ix + 409 + (56 online) pages. https://www.math.brown.edu/johsilve/frint.html

# 1.5 More on this topic

## 01 | Proof: the ultimate truth in mathematics

*How do you know whether something is true or not? Surely, you've been told that the angles of a triangle add to 180°, for example, but how do you know for sure? What if you met an alien who had never studied basic geometry? How could you convince him/her/it that this fact is true? In a way, this is what mathematics is all about: devising new statements, deciding somehow whether they are true or false, and explaining these findings to other people (or aliens, as the case may be).*

*[…] As a side note, you might even wonder "What does it mean for something to be true?" A full discussion of this question would delve into philosophy, psychology, and maybe linguistics, and we don't really want to get into that. The main idea in the context of mathematics, though, is that something is true only if we can show it to be true always. We know 1 + 1 = 2 always and forever. It doesn't matter if it's midnight or noon, we can rest assured that equation will hold true. (Have you ever thought about how to show such a fact, though? It's actually quite difficult! A book called the Principia Mathematica does this from "first principles" and it takes the authors many, many pages to even get to 1 + 1 = 2!) This is quite different from, perhaps, other sciences. If we conduct a physical experiment 10 times and the same result occurs, do we know that this will always happen? What if we do the experiment a million times? A billion? At what point have we actually proven anything? In mathematics, repeated experimentation is not a viable proof! We would need to find an argument that shows why such a phenomenon would always occur. As an example, there is a famous open problem in mathematics called the Goldbach Conjecture. It is unknown, as of now, whether it is true or not, even though it has been verified by computer simulations up until a value of roughly $10^{18}$. That's a huge number, but it is still not enough to know whether the conjecture is True or False. Do you see the difference? We mathematicians like to prove facts, and checking a bunch of values but not all of them does not constitute a proof.*[7]

Proof is therefore what makes mathematics different from other subjects and what characterises mathematics. If we were in ancient Greece, the most recent science would tell us that all things are made up of four elements: fire, water, earth and air. It would also tell us that in a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides. Today we still believe one of these two facts, but not the other. The difference between the two: the Pythagorean theorem has been proved (Hart, 2024). And a theorem that has been proved once in mathematics, even if its proof is thousands of years old, will remain true forever. This is the great difference between mathematics and the other natural sciences.

Once there is a valid proof of a theorem, that's enough, in purely mathematical terms. You don't have to prove it again, and proving it again doesn't make it 'truer'. So why do some results have many proofs? The Pythagorean theorem is an example. With hundreds of proofs, it is perhaps the most proven theorem of all time. There are several reasons for this multiplicity of proofs. Firstly, the theorem has been noted in many different cultures. Even ancient Egypt, which probably didn't know the general result, seems to have known and used the fact that a 3-4-5 triangle has a right angle opposite the longest side. Later, in the Greek, Indian and Chinese traditions, we find different proofs. But even when the proofs of the result were well known, people continued to

[7] This text is the beginning of the book *Everything You Always Wanted To Know About Mathematics* (https://ia600408.us.archive.org/26/items/everything-you-always-wanted-to-know-about-mathematics/bws_book.pdf), which we highly recommend to readers.

find new ones. In the case of the Pythagorean theorem, this is something of a mathematical hobby, but sometimes the discovery of another proof of a result can show previously unnoticed connections or potential generalisations. Sometimes, by using a new technique, we realise that it can be applied more widely. Furthermore, it often happens that the first proof of a theorem is not the best - it does the job- but later people find clearer arguments and shorter paths. Over the years, proofs can be refined and simplified so that eventually a beautiful, crystalline kernel remains, where the exquisite genius of the underlying idea is shown to its best effect. (Hart, 2024). Let us consider problem 09 of this module: at first sight it seems that $n^2 + n + 41$ is always prime for any n. In any case, the expression is a prime for $n = 0,1,2,3,4,5,6,7,8,9,10$. So does it always give a prime number? The answer is no (see *1.7 Solutions*). A first proof of this could be to simply test all the values of n one after the other and realise that at $n = 40$ , the expression gives 1681 which is not prime, because 1681 is divisible by 41. A quick way of seeing this divisibility (without using any technological tools) is the following calculation:

$$40^2 + 40 + 41 = 40 \cdot (40 + 1) + 41 = 40 \cdot 41 + 41 = 41 \cdot (40 + 1) = 41^2.$$

This is certainly a completely correct mathematical proof, but it is not very elegant. A more elegant proof is to see that if we take $n = 41$ , we obtain a sum of 3 terms each of which is divisible by 41 (the sum being $41^2 + 41 + 41$) and so the whole expression is divisible by 41 and therefore does not give a prime number. For more examples of elegant proofs, we recommend that readers watch the lecture *A Mathematician's View of a Proof* given by the British mathematician Sara Hart.

A Mathematician's View

https://www.gresham.ac.uk/watch-now/mathematician-proof

# 02 | Major conjectures proven and still open

" Mathematics is a vast, ever-growing, ever-changing subject. Among the innumerable questions that mathematicians ask, and mostly answer, some stand out from the rest: prominent peaks that tower over the lovely foothills. These are the really big questions, the difficult and challenging problems, that any mathematician would give his or her right arm to solve. Some remained unanswered for decades, some for centuries, a few for millennia. Some have yet to be conquered." So begins the book *The Great Mathematical Problems* written by Ian Stewart (Stewart, 2014).

## Fermat's Last Theorem

One of the best-known problems is *Fermat's Last Theorem*. The French mathematician Pierre de Fermat stated this theorem in a marginal note in his copy of Diophantus' *Arithmetica* in the 17th century. At one point in this book, the Pythagorean triplets are discussed. In the margin, Fermat writes:

*It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.*

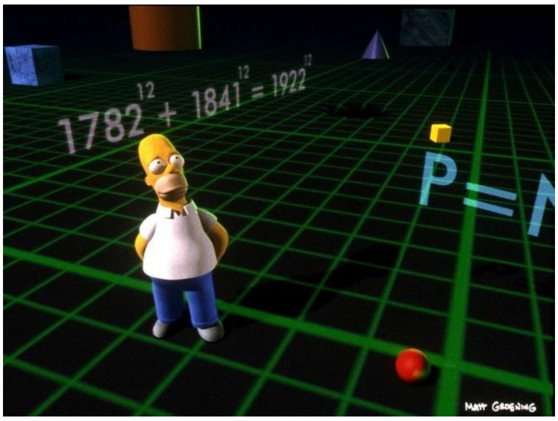
The modern mathematical statement is as follows:

**Fermat's last theorem.** *For any integer $n > 2$ , there is no non-trivial triple of strictly positive integers $(x, y, z)$ such that the following equation is satisfied:*

$$x^n + y^n = z^n.$$

Trivial solutions correspond to cases where one of the terms is zero. For more than three centuries, this theorem remained unproved.

In *the Treehouse of Horrors VI* episode of *The Simpsons*, there is a reference to Fermat's famous theorem.




If the equation behind Homer Simpson were true, a counterexample to Fermat's Last Theorem would have been found. By calculating, we find that the left-hand side is equal to

$$2541210258614589 \ldots$$

while the right-hand side is

$$2541210259314801 \ldots$$

Both numbers have 40 digits and the first 9 digits are the same. However, you can see very quickly, without calculation, that this equation cannot be correct because the left-hand side is an odd number while the right-hand side is an even number.

Many mathematicians have tried to prove Fermat's Last Theorem, but without success. It was not until 1994 that British mathematician Andrew Wiles succeeded in proving the theorem using advanced concepts from number theory and algebraic geometry, in particular elliptic curves and modular forms. The story behind the mathematician Wiles is impressive too. At the age of 10, Andrew Wiles discovered *Fermat's Last Theorem*, a centuries-old mathematical puzzle. This encounter marked a turning point in his life, and he vowed to become a mathematician when he grew up and to solve this mystery one day. During his studies in mathematics, Wiles specialised in number theory, a branch that was home to the famous theorem. However, the more he immersed himself in the mysteries of mathematics, the more he perceived the elusive complexity of the problem. The theorem seemed beyond his grasp, and he temporarily abandoned his ambition. But fate sometimes has surprises in store. Other mathematicians made the connection between *Fermat's Last Theorem* and elliptic curves and modular forms, subjects in which Wiles had meanwhile become an expert. For seven intense and secret years, Wiles devoted himself wholeheartedly to the proof. In 1994, in a series of lectures, Wiles finally revealed his proof to the world. The announcement caused a sensation, even in the mainstream media - a rare event for a mathematical proof.

Interview with Andrew Wiles

https://www.youtube.com/watch?v=nlUimyJpWtI

Thus, *Fermat's Last Theorem*, which had haunted minds for centuries, was finally answered thanks to the tenacity and ingenuity of Andrew Wiles. Wiles' proof has had a profound impact on mathematics. It opened up new perspectives in number theory and strengthened our understanding of elliptic curves and modular forms. *Fermat's Last Theorem* is now an emblematic example of the power of mathematics and the perseverance of researchers (Stewart, 2014). For a more extensive history of *Fermat's Last Theorem*, we recommend Simon Singh's incredible book *Fermat's Last Theorem* (Singh, 2002). A nice summary of the history of Fermat's Last Theorem is also given in the first 30 minutes of the podcast *Dernier Théorème de Fermat : à l'épreuve de l'informatique* from *France Culture*'s series *La Science, CQFD*, mentioned at the end of section *03 And what about artificial intelligence*?

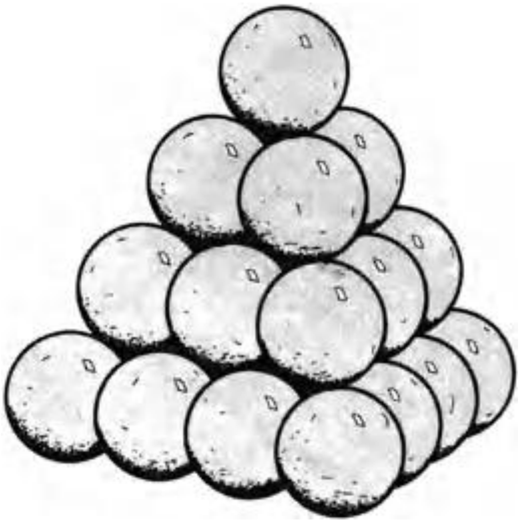
## Kepler's conjecture

Another well-known conjecture that took several hundred years to solve is the Kepler conjecture. In 1611, the astronomer Johannes Kepler, who had begun studying arrangements of spheres, stated his conjecture:

**Kepler's conjecture.** *The close packing is the densest possible sphere packing.*

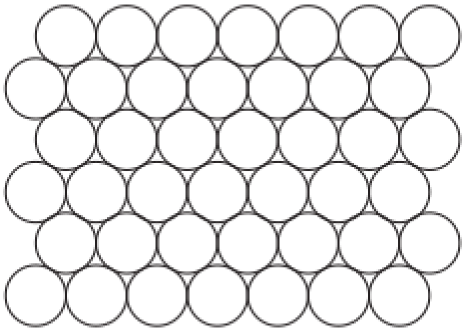
To explain this in a simpler way, let's consider a large empty space, such as an aircraft hangar, and ask ourselves what is the greatest number of balls that can be placed in it. If, instead of balls, we try stacking identical wooden blocks, like children's building blocks, the answer becomes easy. The cubes fit together with no wasted space and we can fill almost one hundred per cent of the space (ignoring the small amount of space that may be left around the walls and ceiling). Hence the number of cubes we can stack is almost equal to the volume of the shed divided by the volume of one of the cubes.

But spheres don't fit together as well as cubes, and there is always wasted space in between. No matter how ingeniously the balls are arranged, about a quarter of the space will go unused. A familiar arrangement is as follows:



Source: Compact stack of compact plans (Conway & Sloane, 1999)

In 1611, Kepler claimed that this arrangement was in fact the most optimal stack in existence. He was convinced of this but was unable to prove it. It was only in 1998 that Thomas Hales announced that he had proved the conjecture. His proof was very long and computer-assisted. It was extraordinarily difficult to verify but is now fully verified and accepted by the mathematical community. The story didn't end there, because the question of an optimal stacking of spheres can arise in any dimension (and not just dimension 3). This more general problem is known as the *Sphere Packing Problem*. The optimal solution in dimension 2 is as follows:



Optimal stacking in dimension 2 was only proved in 1940. For higher dimensions, the problem was recently solved for dimensions 8 and 24 by the Ukrainian mathematician Maryna Viazovska (Klarreich, 2016).

Source : https://www.faz.net/aktuell/wissen/falling-walls/die-mathematikerin-maryna-viazovska-gewinnerin-der-fields-medaille-im-gespraech-18452365.html

For all other dimensions (dimensions strictly greater than 3 and different from 8 and 24), the optimal stacking is not yet known.

La conjecture
de Kepler

https://www.arte.tv/fr/videos/107398-007-A/voyages-au-pays-des-maths/

## The twin primes conjecture

A third conjecture we wanted to talk about is the twin primes conjecture.

**Definition.** *Let $p_1$ and $p_2$ be two prime numbers. We say that $(p_1, p_2)$ forms a pair of twin primes if* $p_2 - p_1 = 2$.

*The first pairs of twin primes are*

$$(3,5), (5,7), (11,13), (17,19), \dots$$

We have known since Euclid (300 BC) that there are an infinite number of prime numbers. The question that then arises, and that mathematicians have been asking since the 19th century, is whether there is also an infinite number of twin primes. As the proof of the infinity of primes is relatively simple, mathematicians initially thought that the proof of the infinitude of twin primes could not be very complicated either. But no one could find a proof. Of course, computers were finding bigger and bigger twin primes. The largest known to date are

$$2996863034895 \cdot 2^{1.290.000} - 1 \text{ and } 2996863034895 \cdot 2^{1.290.000} + 1.$$

This pair was discovered in 2016. We also know that there are 808,675,888,577,436 twin primes with less than 18 digits. All these facts mean that mathematicians are convinced that there is indeed an infinite number of twin primes. However, as we saw in the previous paragraph, being convinced or finding a large number of examples that work does not constitute proof in mathematics. In May 2013, everything changed with the publication of an article by mathematician Yitang Zhang. In the article he proved the following theorem.

**Theorem (Zhang, 2013)**. *There are infinitely many primes $(p_1, p_2)$ such that*

$p_2 - p_1 \leq 70{,}000{,}000.$

This theorem caused a sensation in the mathematical world. Why did it cause such a stir? Until 2013, nothing was known about twin primes. Mathematicians were convinced that there were infinitely many, but they had no proof. So it could be that the prime numbers were getting further and further apart: that after a certain huge number, we would no longer find twin primes, and then after an even bigger number, we would no longer find primes with a distance of 4, and then 6, and so on. This was a real possibility. In 2013, Zhang demonstrated that if we give ourselves the number 70,000,000 as an upper limit, then we will always find primes that are at most 70,000,000 apart as far down the number line as we go. Let's face it, 70,000,000 is not at all equal to 2, but the two numbers aren't that different either, in the sense that Zhang's theorem was a first step in the right direction. In the year that followed, several other mathematicians, including Terence

Tao (see next section), succeeded in reducing the number from 70,000,000 to 246, which gives the current theorem :

**Theorem (Maynard, Polymath, Tao, 2014).** *There are infinitely many primes* $(p_1, p_2)$ *such that* $p_2 - p_1 \leq 246.$

The limit 246 is already closer to 2 than the previous limit, but unfortunately Tao and Maynard have also shown that with the methods applied so far, it will be impossible to reduce this limit any further. So a new idea or new methods will have to be found mathematically, which means that although there has been a breakthrough in the twin primes conjecture, the conjecture remains open.

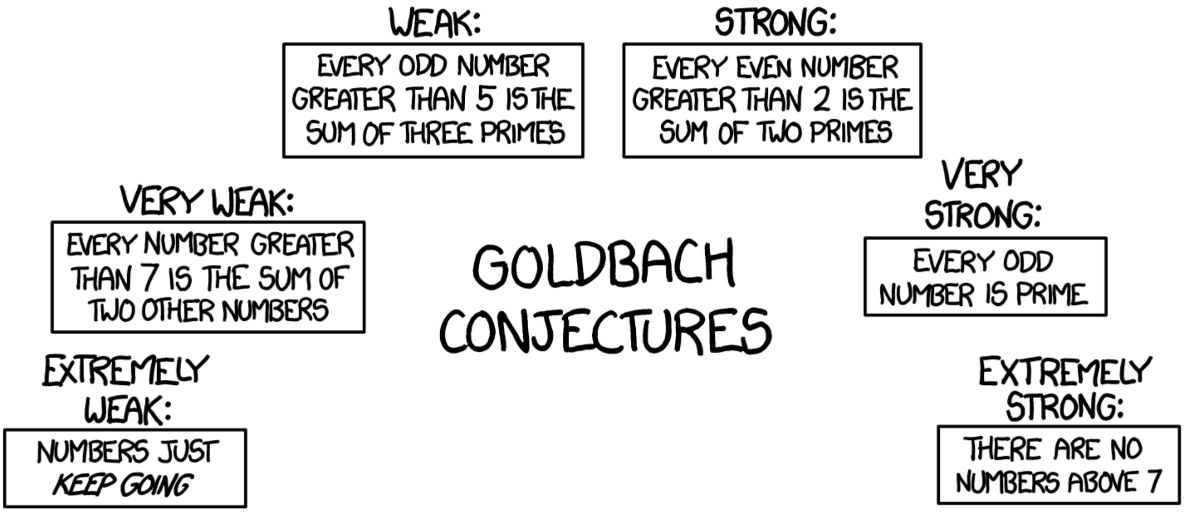
Other well-known open conjectures are those illustrated in problems 13 and 14 of this module.

## The Goldbach conjecture

Problem 13 is known as the *Goldbach conjecture*, formulated in 1742 by Christian Goldbach.

**Goldbach's conjecture.** *Any even integer greater than 3 can be written as a sum of two primes.*

This is one of the oldest open conjectures in mathematics. It is verified for any even number less than $4 \cdot 10^{18}$ , but remains open in general. There is a weak version of Goldbach's conjecture, which states that any odd number greater than 9 can be written as the sum of three primes. This version is called the weak version, because if Goldbach's conjecture is proved one day, the weak version follows straight away: every odd number *a* greater than 9 is written as the sum of an even number *b* and 3. So if *b* is written as the sum of two primes (by Goldbach's conjecture), *a* is written as the sum of these two primes and 3, so 3 primes. In 2013, the Peruvian mathematician Harald Helfgott published a proof of the weak version. This proof has still not been verified.




Source : xkcd https://xkcd.com/1310/

Terence Tao, whom we have already met several times, has proved the following theorem.

**Theorem (Tao, 2012)**. *Any odd integer greater than 1 can be written as a sum of at most five primes.*

Goldbach's conjecture is part of problem 8 of Hilbert's problems, together with the twin primes conjecture and *the Riemann hypothesis. The Riemann hypothesis* is probably the best-known conjecture in mathematics. This seemingly anecdotal hypothesis put forward by Bernhard Riemann one hundred and fifty years ago concerns the distribution of prime numbers. A fascinating and understandable summary is given in the book *The Music of the Primes* by mathematician Marcus du Sautoy (du Sautoy, 2003). The following video provides a very brief summary.

L'hypothèse de Riemann

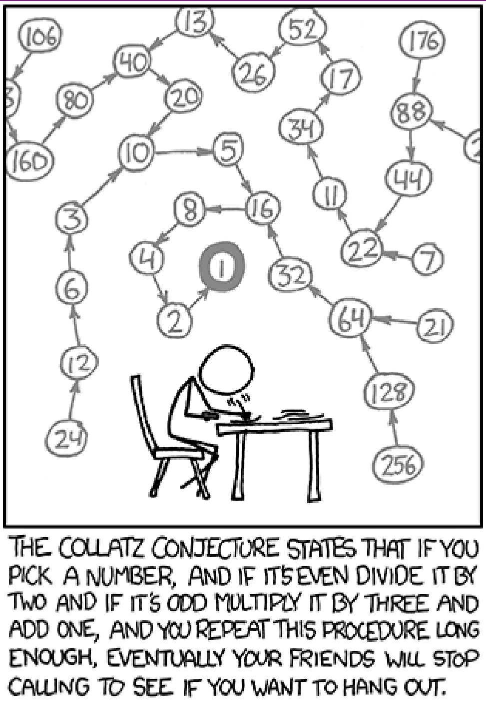
https://www.arte.tv/fr/videos/097454-011-A/voyages-au-pays-des-maths/

At the Second International Congress of Mathematicians, held in Paris in August 1900, David Hilbert presented a list of problems that had hitherto challenged mathematicians. According to Hilbert, these problems were to mark the development of mathematics in the 20th century, and it is undeniable that they had a considerable impact. After the congress, the final list included 23 problems, known today as Hilbert's problems. Today, 8 of these problems are completely solved, 10 have partial solutions and 5 are still unsolved. A funny anecdote is that Hilbert expressed somewhat contradictory opinions about the difficulty of the Riemann hypothesis. At one point he compared three unsolved problems: the transcendence of $2^{\sqrt{2}}$ , *Fermat's Last Theorem* and *the Riemann Hypothesis*. According to him, the Riemann Hypothesis would probably be solved in a few years, Fermat's Last Theorem perhaps in his lifetime, and the transcendence question perhaps never. Surprisingly, the question of transcendence was resolved a few years later. As seen before, Andrew Wiles recently proved Fermat's Last Theorem and Riemann's hypothesis remains open. On another occasion, Hilbert remarked that if he woke up after a sleep of five hundred years, his first question would be whether the Riemann hypothesis had been solved (Conray, 2003).

The 21st century equivalent of Hilbert's problems is the list of seven Millennium Prize problems selected in 2000 by the Clay Institute of Mathematics: https://www.claymath.org/millennium-problems/. Unlike the Hilbert problems, for which the main reward was the admiration of David Hilbert himself and mathematicians in general, each of the prize problems comes with a $1 million prize. To date, only one of the seven problems has been solved. It is the Poincaré conjecture, which has been solved by the eccentric Russian mathematician Grigori Perelman.

## The Collatz conjecture

Problem 14 in this module is known in the mathematical world as the Collatz or Syracuse conjecture. This conjecture is not one of the great mathematical problems, but it has become part of mathematical folklore because, despite the simplicity of its statement (it can be explained to a primary school child), this conjecture has been challenging mathematicians for many years.




Source : xkcd https://xkcd.com/710/

The following video by Derek Muller of the *Veritasium* YouTube channel provides more information.

Collatz Conjecture

https://www.youtube.com/watch?v=094y1Z2wpJg

## The Euler conjecture

Just as in this module, there have also been conjectures in the history of mathematics that mathematicians were convinced were true, but which turned out to be false in the end. A well-known example of this is the Euler conjecture, which was originally proposed by the Swiss mathematician Leonhard Euler in 1772 and which reads as follows:

**Euler's conjecture**. *For any integer n strictly greater than 2, the sum of n - 1 nth powers is not an nth power.*

Note that the case $n = 3$ corresponds to Fermat's Last Theorem with exponent 3. It was only in 1966 that two mathematicians found a counterexample and wrote one of the shortest articles ever published in mathematics:

**COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS**

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [1] that at least $n$ $n$th powers are required to sum to an $n$th power, $n > 2$.

REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.

https://www.ams.org/journals/bull/1966-72-06/S0002-9904-1966-11654-3/S0002-9904-1966-11654-3.pdf

## The Pólya conjecture

A final example of a refuted conjecture is Polya's conjecture[8] . We'll start with a definition.

**Definition**. *A positive integer is called of even type if its prime factorisation contains an even number of prime factors. Otherwise it is called of odd type.*

Let's do an example. Take $15 = 3 \cdot 5$. The number 15 has 2 prime factors and is therefore of even type. On the other hand,$12 = 2 \cdot 2 \cdot 3$ and therefore 12 is of odd type.

By definition, 1 is of even type and all primes are of odd type.

**Definition**. *E(n) denotes the number of positive integers less than or equal to n which are of even type and O(n) the number of positive integers less than or equal to n which are of odd type.*

Let's try to determine $E(9)$ and $O(9)$. Take all the positive integers from 1 to 9 inclusive and decide whether they are of even or odd type.

- The numbers 1, 4, 6 and 9 are of even type. There are 4 in all. So $E(9) = 4$.
- The numbers 2, 3, 5, 7 and 8 are of odd type. There are 5 in all. So $O(9) = 5$.

**Polya conjecture.** For any integer $n \geq 2$ , E(n) is less than or equal to O(n).

Is there a proof for Polya's conjecture?

The answer is ... no!

In 1962, Sherman Lehman found that this conjecture fails for $n = 906\,180\,359$. The smallest counterexample is $n = 906\,150\,257$ and was discovered by Minoru Tanaka in 1980.

Polya's conjecture is therefore false.

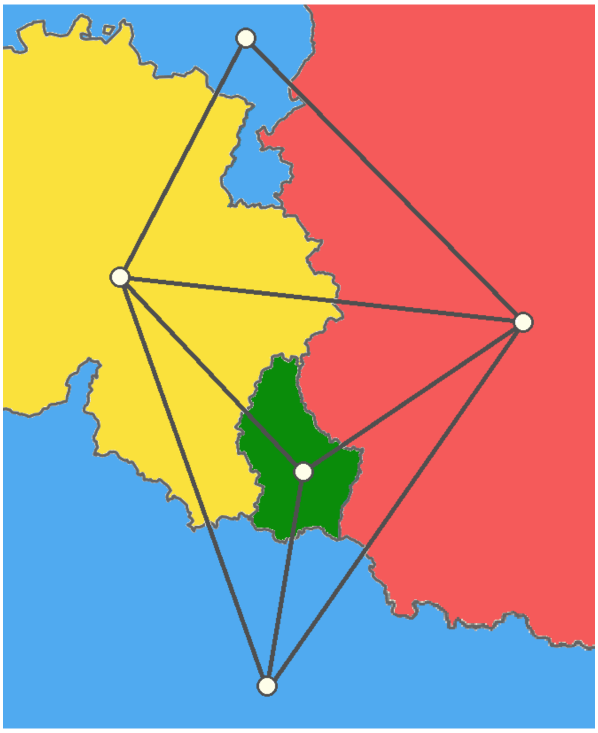
And finally, a quote from Polya about teaching mathematics:

[8] George PÓLYA, Hungarian-born mathematician, 1887-1987

*A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.*

## 03 | What about artificial intelligence?

Proofs involving computer calculations have been around for some time. The most famous is the proof of the four-colour theorem. The four-colour theorem states that it is possible, using only four different colours, to colour any map divided into regions, so that two adjacent regions, i.e. those that have an entire border (and not just a point) in common, always receive two distinct colours.



Source: Wikipedia, Four-colour colouring of five countries

The result was conjectured in 1852 by Francis Guthrie, who was interested in colouring the map of the regions of England. Two early claimed proofs were published, by Alfred Kempe in 1879 and Peter Guthrie Tait in 1880. But they proved to be wrong; the errors were only identified in 1890.

Although Kempe's proof turned out to be false, it does provide a proof of a similar problem, with five colours instead of four, known today as the five-colour theorem.

In 1976, two Americans, Kenneth Appel and Wolfgang Haken, claimed to have demonstrated the four-colour theorem. Their proof shocked the scientific community: for the first time, a proof required the use of a computer. The two mathematicians had succeeded in reducing the infinite number of possible maps to exactly 1,482. To check that these 1,482 cases could all be coloured with just 4 colours, they wrote a computer program that tested the 1,482 cases one by one (over 1,200 hours of calculation). Since 1976, other computer programs, written independently of the first, have produced the same result.

To this day, no proof has been discovered that can do without a computer; however, many enthusiasts continue to be convinced that they have proved the Four Colour Theorem without a computer. In the book *Mathematical Cranks (Underwood, 1992)*, a whole chapter is devoted to these sometimes very funny attempts.

So how do we know that the proof of the Four Colour Theorem is correct? Are these computer proofs valid? For most mathematicians, yes. The reason is that, as long as we check the program, the risk of a stray neutrino turning a 0 into a 1 and producing an erroneous result seems smaller than the risk of there being an error in one of the hundreds of pages of hand-written proofs (such as Fermat's theorem). Human beings are considerably more fallible than machines, if we correctly asked the machine to do what we wanted it to do. Proofs done by hand, where anyone (with sufficient mathematical knowledge) can check the proof line by line, are of course still preferable, but such proofs are sometimes hard to find. In this case a computer proof is better than no proof (Hart, 2024).

Now, with recent advances in artificial intelligence, the question arises as to whether computers will soon be able to prove theorems on their own, without the help of a human being. According to Fields Medal winner Terence Tao, we're not there yet. In an interview with *Quanta Magazine*, he states that artificial intelligence tools are not yet capable of doing the high-level mathematics of current research. However, he is convinced that it will come (Strogatz, 2024).

*I think, the next cultural shift will be whether AI-generated proofs will be accepted. Right now, AI tools are not at the level where they can generate proofs to really advance mathematical problems. Maybe undergraduate-level homework assignments, they can kind of manage, but research mathematics, they're not at that level yet. But at some point, we're going to start seeing AI-assisted papers come out and there will be a debate. - Terence Tao, 2024.*

There are currently projects aimed at formalising the proofs of major theorems in what are known as *formal proof assistants*. These are computer languages capable of checking whether a proof is true or not, and therefore whether the theorem has been proved or not (Strogatz, 2024). These assistants will enable many more mathematicians to work together. For the moment, a collaboration on a project never contains more than 5 mathematicians. In a collaboration, you either have to trust the other mathematicians completely, or check all their calculations and proofs line by line. With more than 5 people, this becomes too chaotic. But once these *formal proof assistants* are in place, much larger collaborations are entirely possible, because these tools will check the different steps (Drösser, 2024).

An example of such an assistant is given in the podcast *Dernier Théorème de Fermat : à l'épreuve de l'informatique* from *France Culture*'s series *La Science, CQFD*. The podcast talks about a proof assistant that is verifying the proof of Fermat's Last Theorem mentioned earlier. The first 30 minutes of the podcast provide a nice summary of the history of Fermat's Last Theorem:

La Science,
CQFD

Link to the podcast

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# 1.6 A word from the scientists

## 01 Mäin Element - Hugo Parlier

Hugo Parlier, currently a professor in the mathematics department at the University of Fribourg, is a mathematician whose career is a perfect illustration of the synergy between academic research and science outreach.

A graduate of the École Polytechnique Fédérale de Lausanne, where he obtained his doctorate in 2004, Hugo Parlier broadened his experience through several postdoctoral stays in Madrid, Toronto and Geneva. His career then took him to the University of Luxembourg (2017-2025), before his recent moving to Fribourg. His research focuses on a variety of areas, including geometry, topology and combinatorics.

What sets Hugo Parlier apart is his ongoing commitment to making mathematics accessible to the general public. In particular, he is a co-creator of Mathema, an interactive book for iPad that gave rise to puzzles called Quadratis, available on iOS.

His involvement in science outreach has been demonstrated through several large-scale projects: *The Simplicity of Complexity* and *Recreate*: *shapes from the collective Imagination*, which were presented at the Dubai World Expo in 2021-2022; *The Sound of Data*, as part of Esch2022, European Capital of Culture; the exhibition *Shapes : Patterns in Art and Science* at the EPFL Pavillons, a collaborative project exploring the links between art and science.

To find out more about his work and his vision of mathematics, you can listen to his interview in the Mäin Element podcast, a collaboration between the Lëtzebuerger Journal and the Fonds National de la Recherche (FNR).

Mäin Element,
Hugo Parlier

https://journal.lu/fr/main-element-hugo-parlier

# 02 Paroles de Déchiffreurs

What does it mean to be a mathematician? How do mathematicians work? What is intuition? Where do their ideas take them? What is a researcher's day like? These and many other questions were put to fourteen researchers working in different areas of mathematics in interviews that make up the web documentary "Paroles de déchiffreurs". A physicist, an astronomer and a historian of mathematics also contributed their answers.

Sprinkled with doubts, joys, failures, questions, desires... the stories of these researchers reveal a world little known to the general public. They all demonstrate a great passion for their work and show how, without their great capacity for work and concentration, they would not have got to where they are today: on the frontier of the latest mathematical discoveries. (imagesdesmaths, 2024)

Between 2011 and 2012, an exhibition of photographs entitled *Les Déchiffreurs* toured the Hauts-de-France region. This exhibition, created by the IHÉS in collaboration with photographers Anne Papillault and Jean-François Dars, has evolved into a more ambitious project running until 2018-2019.

The project has evolved to include interviews with mathematicians and physicists, giving rise to a web documentary. Directed by Michaël Mensier from the University of Lille's Audiovisual Department, the project aims to raise awareness of fundamental research in mathematics and physics among the general public and students.

Although the pandemic interrupted the completion of four interviews, the project represents an important testimony to the history of contemporary mathematics.

The web documentary is available at the following address:

Paroles de Déchiffreurs

https://parolesdedechiffreurs.univ-lille.fr/

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